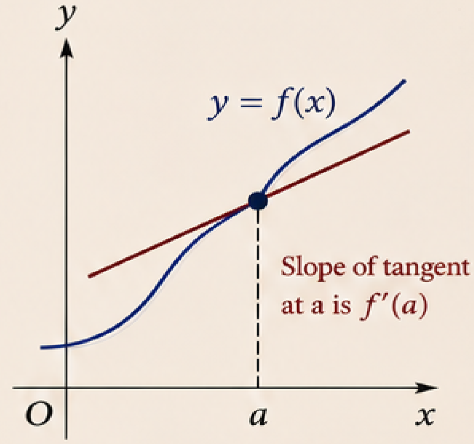
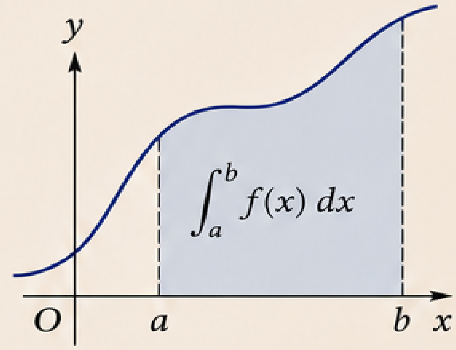
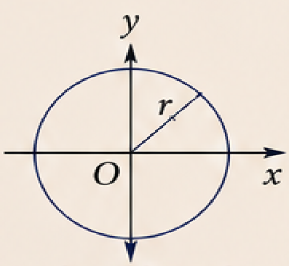
ANALYTIC GEOMETRY & CALCULUS



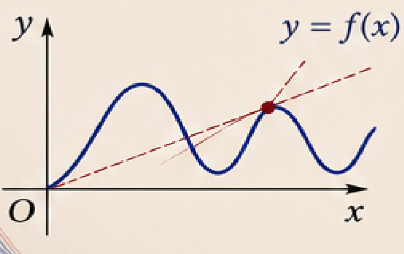
Lecture Notes • Worked Examples • Practice Problems with Solutions

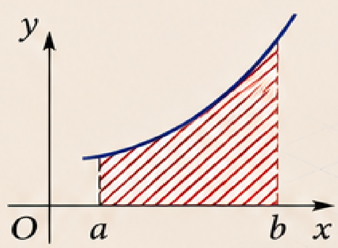


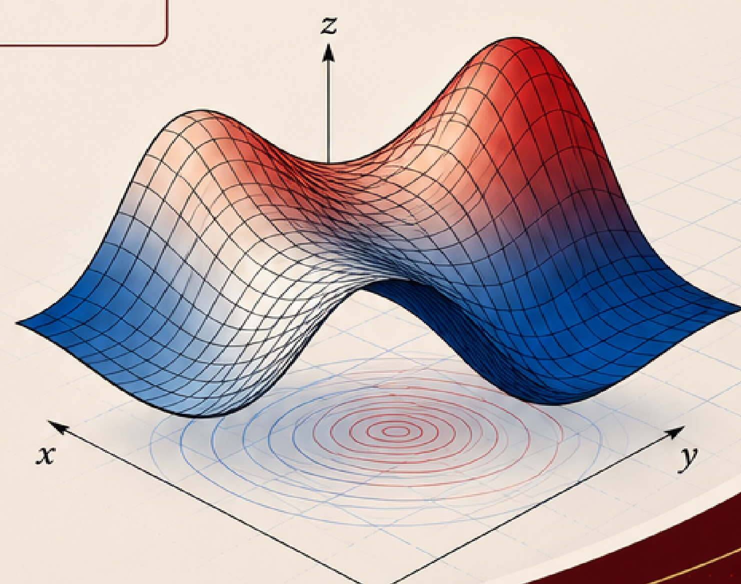


$x^2 + y^2 = r^2$

$$\boxed{\int_a^b f'(x)\,dx = f(b) - f(a)}$$







$$\frac{d}{dx}(x^n) = nx^{n-1}$$

Diyath Nelaka Pannipitiya

ANALYTIC GEOMETRY
&
CALCULUS

Lecture Notes, Worked Examples, and Practice Problems

with Solutions

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About This Book

Version 1.0
July 2026

If you discover any typographical errors, mathematical errors, or have suggestions for improving future editions, I would greatly appreciate hearing from you.

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The most recent version of this book is always available from the website above.

Preface

These notes were developed for MATH-I 165: Analytic Geometry & Calculus I, which I taught during the Summer of 2026 at Indiana University Indianapolis. During the preparation of lectures for the course, the notes gradually expanded into this manuscript.

One concern that many of my students shared with me was how difficult it can be to find concise calculus notes that contain detailed explanations, fully worked examples, and representative practice problems with complete solutions. Although standard calculus textbooks provide excellent coverage of the subject, they are naturally much broader than what can be covered in a single semester. I therefore decided to organize my lecture notes into a resource that focuses on the material most relevant to a first course in calculus.

This book is not meant to replace a comprehensive calculus textbook. Instead, it is designed to serve as a companion resource that emphasizes the fundamental ideas, provides detailed worked examples, and offers additional practice problems with solutions to help students build confidence and develop a solid understanding of calculus.

Since these notes will continue to evolve through classroom use and reader feedback, I welcome suggestions for improving future editions.

I sincerely hope that it will be a valuable resource for students and will help them learn, appreciate, and enjoy calculus.

Diyath Nelaka Pannipitiya
Indianapolis, Indiana
July 27, 2026

Contents

A Note to the Student

Learning calculus can be challenging, especially when the ideas seem unfamiliar at first. Do not become discouraged if a topic does not make sense immediately. Mathematics is a subject that rewards patience and persistence. Every theorem and every technique in this book was once new to every mathematician who learned it.

Calculus is learned by doing mathematics. I encourage you to attempt every example and exercise before reading the solution. Mistakes are a natural part of learning, and each one is an opportunity to deepen your understanding.

Thinking Like a Mathematician

Whenever you solve a problem, do not stop after obtaining the answer. Always ask yourself what you have learned from it.

If you become stuck, do not be discouraged. Sometimes the best decision is to step away from one approach and try another. Progress in mathematics often comes from changing your perspective rather than working harder on the same idea.

Ask yourself:

- Why does this method work?
- Could the same idea solve a different problem?
- What assumptions did I use?
- Is there a shorter or more elegant solution?

Developing the habit of asking these questions will strengthen both your problem-solving skills and your mathematical intuition.

Chapter 1

Algebra and Trigonometry Review

1.1. Goals of This Review

- Algebraic manipulation
- Factoring expressions
- Rational expressions
- Simplifying radicals
- Graph interpretation
- Function notation
- Basic trigonometric identities
- Unit circle values
- Solving basic algebraic equations, inequalities, and trigonometric equations

Important

Why is this important?
Many limit problems in calculus require algebraic simplification and trigonometric identities before the limit can actually be evaluated.

1.2. Review of Functions

What is a function?

A function is a map (or rule) from one set (this input set is called **the domain** of the function) to another set (this output set is called **the range** of the function) such that *each input* has *exactly one output*. The input variable is called the **independent variable** and the output variable is called the **dependent variable**.

1.3. Ways to Represent a Function

A function can be represented in several different ways. In calculus, it is important to understand how to move between these representations.

The four common ways are:

1. By a table
2. By ordered pairs (2-tuples)
3. By a graph
4. By an equation

Important

Different representations describe the same relationship between inputs and outputs. In calculus, we often move between these forms depending on the problem.

Representing a Function by a Table

A table lists selected input and output values.

Example 1.1.

Consider the table given below.

(a) What is the domain of f?

(b) What is the range of f?

x	$f(x)$
-2	4
-1	1
0	0
1	1
2	4

Solution.

(a) $\text{Dom}(f) = \{-2, -1, 0, 1, 2\}$.

(b) $\text{Range}(f) = \{0, 1, 4\}$.

Example 1.2.

The following table represents a function:

x	$g(x)$
1	3
2	5
3	7
4	9

Find:

(a) $g(2)$

(b) $g(4)$

Solution.

(a) $g(2) = 5$

(b) $g(4) = 9$

Representing a Function by Ordered Pairs

A function can also be represented as a set of ordered pairs.

$$\{(x, y)\}$$

where the first coordinate is the input and the second coordinate is the output.

Warning

A relation is a function only if each input has exactly one output. Therefore, if a relation assigns a single input to more than one output, it cannot be called a function. In other words, all functions are relations/maps, but not all relations/maps qualify as functions.

Example 1.3.

Consider the set $\{(-1, 2), (0, 1), (1, 2), (2, 5)\}$.
This means:

$$f(-1) = 2,\ f(0) = 1,\ f(1) = 2,\ f(2) = 5.$$

(a) What is the domain of f?

(b) What is the range of f?

Solution.

(a) $\text{Dom}(f) = \{-1, 0, 1, 2\}$.

(b) $\text{Range}(f) = \{1, 2, 5\}$.

Example 1.4.
Determine whether the following relation is a function:

$$\{(1, 2), (2, 5), (1, 7)\}$$

Solution.
The input 1 is paired with two different outputs:

$$1 \to 2 \quad \text{and} \quad 1 \to 7$$

Therefore, this relation is **not** a function.

Representing a Function by a Graph

A graph visually shows the relationship between inputs and outputs.

Important

A graph represents a function if it passes the Vertical Line Test.

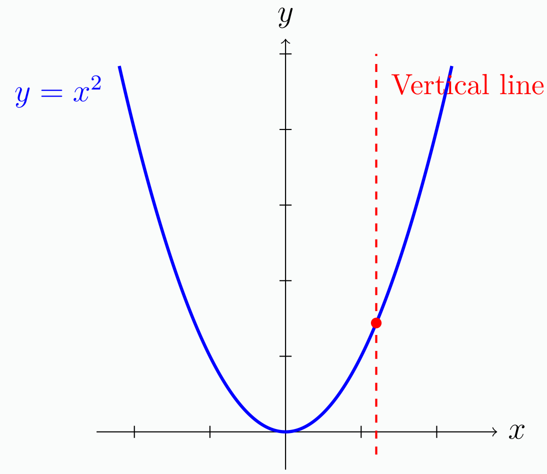
Vertical Line Test

A graph is a function if every vertical line intersects the graph at most once.

Example 1.5.
The graph of

$$y = x^2$$

is a function because every vertical line intersects the graph only once.



Example 1.6.

The graph of

$$x^2 + y^2 = 1$$

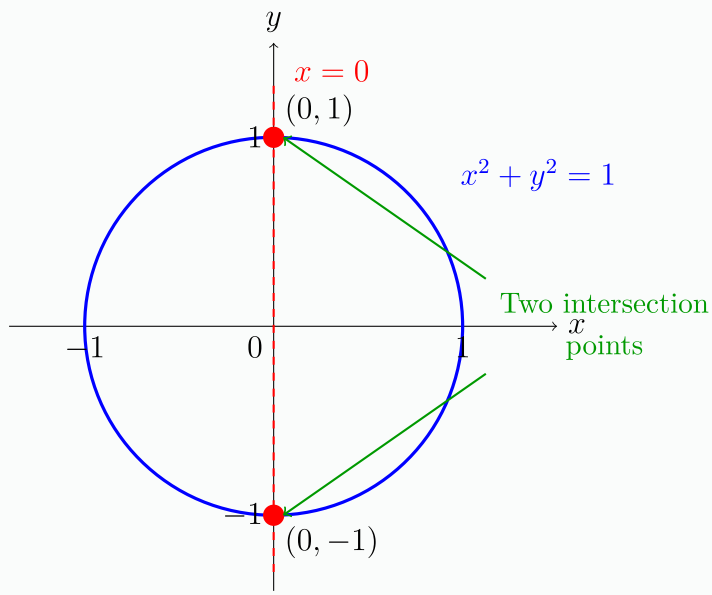
is **not** a function because some vertical lines intersect the graph twice.

For example, when

$$x = 0,$$

we get

$$y = 1 \quad \text{and} \quad y = -1.$$



Representing a Function by an Equation

Many functions are represented by formulas or equations.

Example 1.7.
Consider

$$f(x) = 2x + 1.$$

Find:

(a) $f(3)$

(b) $f(-2)$

Solution.

(a) $f(3) = 2(3) + 1 = 6 + 1 = \boxed{7}$.

(b) $f(-2) = 2(-2) + 1 = -4 + 1 = \boxed{-3}$.

1.4. The Domain and the Range of a Function

Domain of a Function

Common restrictions:

- Denominators cannot equal zero.
- Expressions inside even roots must be nonnegative.

Range of a Function

This is the set of all outputs. That is, if y is an element in the range, then there exists an input x such that y is the output corresponding to the input x.

Example 1.8.
Find the domain and the range of

$$f(x) = \frac{x+2}{x-5}.$$

Write your answer in both set-builder notation and interval notation. Also, graph your answer on the real number line.

Solution.

Domain:
The denominator cannot be zero. Thus,

$$x - 5 \neq 0.$$

Hence,

$$x \neq 5.$$

Therefore, in set-builder notation,

$$\boxed{\{x \in \mathbb{R} : x \neq 5\}}$$

and in interval notation,

$$\boxed{(-\infty, 5) \cup (5, \infty)}.$$

Range:
Let

$$y = \frac{x+2}{x-5}.$$

By solving

$$y(x-5) = x+2 \qquad \text{for} \qquad x$$

we get

$$x = \frac{5y+2}{y-1}.$$

This expression is defined provided

$$y - 1 \neq 0.$$

Thus,

$$y \neq 1.$$

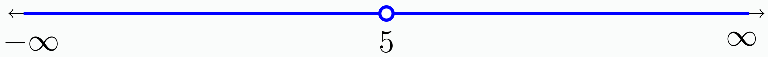
Therefore, in set-builder notation,

$$\boxed{\{y \in \mathbb{R} : y \neq 1\}}$$

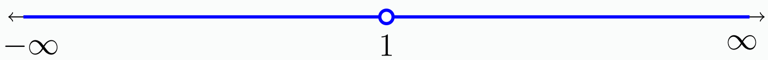
and in interval notation,

$$\boxed{(-\infty, 1) \cup (1, \infty)}.$$

Domain on the Real Number Line



Range on the Real Number Line



1.5. Factoring Polynomials

Factoring is extremely important for simplifying limit expressions.

Common Factoring Patterns

1. Greatest Common Factor (GCF)
2. Difference of squares
3. Trinomials
4. Grouping

Difference of Squares

$$a^2 - b^2 = (a - b)(a + b)$$

Example 1.9.
Factor:

$$x^2 - 25$$

Solution.

$$x^2 - 25 = x^2 - 5^2 = (x - 5)(x + 5).$$

$$\boxed{x^2 - 25 = (x - 5)(x + 5).}$$

Factoring Trinomials

Example 1.10.
Factor:

$$x^2 + 5x + 6$$

Solution.

$$x^2 + 5x + 6 = x^2 + 2x + 3x + 6 = x(x + 2) + 3(x + 2) = (x + 2)(x + 3).$$

$$\boxed{x^2 + 5x + 6 = (x + 2)(x + 3).}$$

Factoring by Grouping

Example 1.11.
Factor:

$$x^3 + 2x^2 + 3x + 6$$

Solution.

$$x^3 + 2x^2 + 3x + 6 = x^2(x + 2) + 3(x + 2) = (x + 2)(x^2 + 3).$$

$$\boxed{x^3 + 2x^2 + 3x + 6 = (x + 2)(x^2 + 3).}$$

1.6. Solving Equations

Example 1.12.
Solve for x:

$$x^2 + 5x + 6 = 0.$$

Solution. From Example 1.10, we have $x^2 + 5x + 6 = (x + 2)(x + 3)$. Thus,

$$(x + 2)(x + 3) = 0.$$

Therefore,

$$\boxed{x = -3 \qquad \text{and} \qquad x = -2.}$$

The Quadratic Formula

Suppose $ax^2 + bx + c = 0$. Then,

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

Example 1.13.
Solve for x using the quadratic formula.

$$x^2 + 5x + 6 = 0.$$

Solution. Here,

$$a = 1, \quad b = 5, \quad c = 6.$$

Thus,

$$\begin{aligned} x &= \frac{-5 \pm \sqrt{5^2 - 4(1)(6)}}{2(1)} \\ &= \frac{-5 \pm \sqrt{25 - 24}}{2} \\ &= \frac{-5 \pm 1}{2} \\ &= \frac{-5 - 1}{2} \qquad \text{or} \qquad \frac{-5 + 1}{2}. \end{aligned}$$

$$\boxed{x = -3, -2.}$$

1.7. Solving Inequalities

- $|x - a| < b$
$$-b + a < x < b + a$$
- $|x - a| \leq b$
$$-b + a \leq x \leq b + a$$
- $|x - a| > b$
$$x < -b + a \quad \text{or} \quad x > b + a$$
- $|x - a| \geq b$
$$x \leq -b + a \quad \text{or} \quad x \geq b + a$$

Example 1.14.
Solve for x:
$$|3x-2|<7.$$
Write your answer in the interval notation.
Solution.
We have
$$\begin{aligned}-7+2 &< 3x < 7+2\\ -5 &< 3x < 9\\ -\frac{5}{3} &< x < 3.\end{aligned}$$
Thus,
$$\boxed{\left(-\frac{5}{3},3\right).}$$

1.8. Rational Expressions

Simplifying Rational Expressions

Important

Always factor completely before canceling.

Example 1.15.
Simplify:
$$\frac{x^2-9}{x^2+5x+6}$$
Solution.
$$\begin{aligned}\frac{x^2-9}{x^2+5x+6} &= \frac{(x-3)(x+3)}{(x+2)(x+3)}\\ &= \frac{(x-3)}{(x+2)}.\end{aligned}$$
$$\boxed{\frac{x^2-9}{x^2+5x+6}=\frac{x-3}{x+2}.}$$

Complex Fractions

Example 1.16.
Simplify:

$$\frac{\frac{1}{x}+\frac{1}{2}}{\frac{1}{x}}$$

Solution.

$$\begin{aligned}\frac{\frac{1}{x}+\frac{1}{2}}{\frac{1}{x}} &= \frac{\frac{2+x}{2x}}{\frac{1}{x}} \\ &= \frac{2+x}{2x}\cdot\frac{x}{1} \\ &= \frac{2+x}{2}.\end{aligned}$$

$$\boxed{\frac{\frac{1}{x}+\frac{1}{2}}{\frac{1}{x}} = \frac{2+x}{2}.}$$

1.9. Radicals and Rationalization

Exponent Rules

- $a^m a^n = a^{m+n}$
- $\dfrac{a^m}{a^n} = a^{m-n}$
- $(a^m)^n = a^{mn}$

Radicals

$$\sqrt{a^2} = |a|$$

A Notation

$$\sqrt[n]{x} = x^{\frac{1}{n}}.$$

Example 1.17.
Simplify:

$$\sqrt{50}$$

Solution.
Because

$$50 = 2 \cdot 5^2,$$

we have

$$\begin{aligned}\sqrt{50} &= \sqrt{2 \cdot 5^2} \\ &= \sqrt{2} \cdot \sqrt{5^2} \\ &= \sqrt{2} \cdot 5\end{aligned}$$

$$\boxed{\sqrt{50} = 5\sqrt{2}.}$$

Rationalizing the Denominator

When a fraction has a radical in its denominator, we often rewrite it in an equivalent form with a rational denominator. This process is known as **rationalizing the denominator**. Depending on the denominator, we multiply by either the radical itself or its conjugate.

Example 1.18.
Simplify:

$$\frac{3}{\sqrt{5}}$$

Solution.

$$\begin{aligned}\frac{3}{\sqrt{5}} &= \frac{3}{\sqrt{5}} \cdot 1 \\ &= \frac{3}{\sqrt{5}} \cdot \frac{\sqrt{5}}{\sqrt{5}} \\ &= \frac{3\sqrt{5}}{\sqrt{5} \cdot \sqrt{5}} \\ &= \frac{3\sqrt{5}}{5}.\end{aligned}$$

$$\boxed{\frac{3}{\sqrt{5}} = \frac{3\sqrt{5}}{5}.}$$

Using Rational Conjugates

Example 1.19.

Simplify:

$$\frac{\sqrt{x}+1}{\sqrt{x}-1}$$

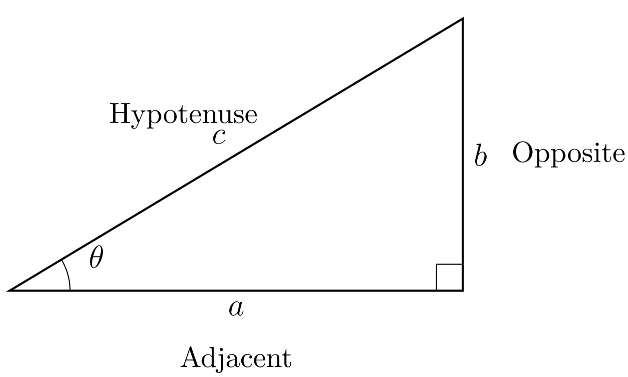
Solution.

$$\begin{aligned}\frac{\sqrt{x}+1}{\sqrt{x}-1} &= \frac{\sqrt{x}+1}{\sqrt{x}-1} \cdot \frac{\sqrt{x}+1}{\sqrt{x}+1}\\ &= \frac{(\sqrt{x}+1)^2}{(\sqrt{x}-1)(\sqrt{x}+1)}\\ &= \frac{(\sqrt{x}+1)^2}{(\sqrt{x})^2-1^2}\\ &= \frac{(\sqrt{x}+1)^2}{x-1}.\end{aligned}$$

$$\boxed{\frac{\sqrt{x}+1}{\sqrt{x}-1}} = \frac{(\sqrt{x}+1)^2}{x-1}.$$

1.10. Trigonometry

Right Triangle Definitions



- $\sin\theta = \dfrac{\text{Opposite}}{\text{Hypotenuse}} = \dfrac{b}{c}$
- $\cos\theta = \dfrac{\text{Adjacent}}{\text{Hypotenuse}} = \dfrac{a}{c}$
- $\tan\theta = \dfrac{\text{Opposite}}{\text{Adjacent}} = \dfrac{b}{a}$
- $\csc\theta = \dfrac{\text{Hypotenuse}}{\text{Opposite}} = \dfrac{c}{b}$
- $\sec\theta = \dfrac{\text{Hypotenuse}}{\text{Adjacent}} = \dfrac{c}{a}$
- $\cot\theta = \dfrac{\text{Adjacent}}{\text{Opposite}} = \dfrac{a}{b}$

Important Trigonometric Identities

- $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$
- $\csc\theta = \dfrac{1}{\sin\theta}$
- $\sec\theta = \dfrac{1}{\cos\theta}$
- $\cot\theta = \dfrac{1}{\tan\theta}$

Formula

Pythagorean identities in trigonometry:

$$\sin^2\theta + \cos^2\theta = 1.$$

$$\sec^2\theta = 1 + \tan^2\theta.$$

$$\csc^2\theta = 1 - \cot^2\theta.$$

Unit Circle Values

Angle	$\sin\theta$	$\cos\theta$	$\tan\theta$
0	0	1	0
$\frac{\pi}{6}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{3}}$
$\frac{\pi}{4}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	1
$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$
$\frac{\pi}{2}$	1	0	Undefined

Radians and Degrees

$$180^\circ = \pi \text{ radians}$$

Example 1.20.

Convert 60° to radians.

Solution.

$$60^\circ = 60^\circ \times \frac{\pi}{180^\circ} = \frac{\pi}{3}.$$

$$\boxed{60^\circ = \frac{\pi}{3}.}$$

Example 1.21.

Convert $\frac{3\pi}{4}$ into degrees.

Solution.

$$\frac{3\pi}{4} = \frac{3}{4} \times 180^\circ = 135^\circ.$$

$$\boxed{\frac{3\pi}{4}} = 135^\circ.$$

1.11. Trigonometric Equations

General Solution for Basic Trigonometric Equations

- If $\sin x = \sin \alpha$, then $x = n\pi + (-1)^n \alpha$, where n is an integer.
- If $\cos x = \cos \alpha$, then $x = 2n\pi \pm \alpha$, where n is an integer.
- If $\tan x = \tan \alpha$, then $x = n\pi + \alpha$, where n is an integer.

Important

Although these are the general solutions, sometimes we only need the solutions within a given interval or range of values.

Example 1.22.

Find the general solution to the equation

$$\sin x = \frac{1}{2}.$$

Solution.

We know that

$$\sin\left(\frac{\pi}{6}\right) = \frac{1}{2}.$$

Thus,

$$\begin{aligned} \sin x &= \frac{1}{2} \\ \sin x &= \sin\left(\frac{\pi}{6}\right) \\ x &= n\pi + (-1)^n \frac{\pi}{6}, \quad \text{where} \quad n \quad \text{is an integer.} \end{aligned}$$

$$\boxed{x = n\pi + (-1)^n \frac{\pi}{6}, \qquad n \in \mathbb{Z}.}$$

Example 1.23.

Solve:

$$\sin x = \frac{1}{2}$$

for $0 \le x < 2\pi$.

Solution.

From the previous example we have the general solution to the equation:

$$x = n\pi + (-1)^n \frac{\pi}{6}, \qquad n \in \mathbb{Z}.$$

In this example, we are interested only in the solutions within the interval $[0, 2\pi]$.
When $n = -1$:

$$x = (-1)\pi + (-1)^{-1}\frac{\pi}{6} = -\pi - \frac{\pi}{6} = -\frac{7\pi}{6} \notin [0, 2\pi].$$

When $n = 0$:

$$x = (0)\pi + (-1)^0\frac{\pi}{6} = 0 + \frac{\pi}{6} = \frac{\pi}{6} \in [0, 2\pi].$$

When $n = 1$:

$$x = (1)\pi + (-1)^1\frac{\pi}{6} = \pi - \frac{\pi}{6} = \frac{5\pi}{6} \in [0, 2\pi].$$

When $n = 2$:

$$x = (2)\pi + (-1)^2\frac{\pi}{6} = 2\pi + \frac{\pi}{6} = \frac{13\pi}{6} \notin [0, 2\pi].$$

Thus, the only solutions in the interval $[0, 2\pi]$ ar

$$\boxed{x = \frac{\pi}{6}, \frac{5\pi}{6}.}$$

Exercises

1. Determine whether each relation is a function.

   (a) $\{(1,2),(2,3),(3,4)\}$

   (b) $\{(-1,4),(0,2),(-1,5)\}$

2. Find the domain and the range of the following relation represented by the ordered pairs:
$$\{(-2,3),(-1,1),(0,0),(2,3),(4,5)\}.$$

3. Evaluate the function
$$f(x) = 3x^2 - 2x + 1.$$
   Find

   (a) $f(2)$

   (b) $f(-1)$

   (c) $f(0)$

4. Find the domain of each function.

   (a) $f(x) = \dfrac{x+4}{x-3}$

   (b) $g(x) = \sqrt{x-7}$

   (c) $h(x) = \dfrac{\sqrt{x+1}}{x-2}$

5. Find both the domain and the range of
$$f(x) = \frac{2x-1}{x+3}.$$
   Write your answers in

   (a) set-builder notation,

   (b) interval notation.

6. Factor completely.

   (a) $x^2 - 49$

   (b) $x^2 + 9x + 20$

   (c) $2x^2 + 7x + 3$

(d) $x^3 - 4x^2 + 5x - 20$

7. Solve.

(a) $x^2 - 7x + 12 = 0$

(b) $2x^2 + 5x - 3 = 0$

(c) $x^2 + 4x + 7 = 0$

8. Solve the inequalities. Write your answers in interval notation.

(a) $|2x - 5| < 7$

(b) $|3x + 1| \geq 4$

9. Simplify each rational expression.

(a) $\dfrac{x^2 - 16}{x^2 + x - 20}$

(b) $\dfrac{x^2 + 7x + 12}{x^2 - 16}$

10. Simplify each complex fraction.

(a) $\dfrac{\frac{1}{x} + \frac{1}{3}}{\frac{1}{x}}$

(b) $\dfrac{\frac{2}{x} - \frac{1}{5}}{\frac{1}{x}}$

11. Simplify.

(a) $\sqrt{72}$

(b) $\sqrt{147}$

(c) $\sqrt{48x^2}$

12. Rationalize the denominator.

(a) $\dfrac{5}{\sqrt{3}}$

(b) $\dfrac{4}{2 + \sqrt{5}}$

(c) $\dfrac{\sqrt{7}}{\sqrt{2}}$

13. Simplify using exponent rules.

(a) $a^5 a^3$

(b) $\dfrac{x^8}{x^3}$

(c) $(2y^4)^3$

(d) $x^{1/2}x^{3/2}$

14. Evaluate the following exactly.

    (a) $\sin\left(\frac{\pi}{6}\right)$

    (b) $\cos\left(\frac{\pi}{3}\right)$

    (c) $\tan\left(\frac{\pi}{4}\right)$

    (d) $\sin\left(\frac{\pi}{2}\right)$

15. Convert.

    (a) $45°$ to radians

    (b) $225°$ to radians

    (c) $\dfrac{7\pi}{6}$ to degrees

    (d) $\dfrac{5\pi}{3}$ to degrees

16. Verify the identity.

    (a) $\dfrac{\sin x}{\cos x} = \tan x$

    (b) $1 + \tan^2 x = \sec^2 x$

17. Find the general solution.

    (a) $\sin x = \frac{\sqrt{3}}{2}$

    (b) $\cos x = -\frac{1}{2}$

    (c) $\tan x = 1$

18. Solve each equation for
$$0 \leq x < 2\pi.$$

    (a) $\sin x = \frac{\sqrt{3}}{2}$

    (b) $\cos x = -\frac{1}{2}$

    (c) $\tan x = -1$

19. Decide whether each graph represents a function. If not, explain why using the Vertical Line Test.

(a) $y = |x|$

(b) $x = y^2$

(c) $y = \sqrt{x}$

20. Challenging Problem.

    Find the domain of

$$f(x) = \frac{\sqrt{x-1}}{x^2 - 9}.$$

Write your answer in both set-builder notation and interval notation.

Chapter 2

The Limit of a Function

2.1. The Limit of a Function

Limits describe the behavior of a function as the input approaches a certain value. At first, limits may seem like an unnecessary abstraction. Why not simply substitute the value of the variable into the function?

The answer is that direct substitution does not always work. For example, some functions are undefined at the very point where we want to understand their behavior.

Limits allow us to describe what happens *near* a point rather than exactly at the point itself. This idea is the foundation of continuity, derivatives, and integrals. Without limits, calculus would not exist.

The Idea of a Limit

Suppose a function $f(x)$ is defined near a number a. When we write

$$\lim_{x \to a} f(x) = L,$$

we mean that the values of $f(x)$ get closer and closer to L as x gets closer and closer to a.

Important

The value of the limit depends on what happens *near* a, not necessarily at a itself.

Example Using a Table of Values

Consider the function $f(x) = \dfrac{x-2}{x^2-4}$.

Let's investigate what happens (to $f(x)$) as x gets closer and closer to 2. The following table provides output values ($f(x)$) for some input values (x) near 2.

Example 2.1.

x	$f(x)$
1.9	0.25641
1.99	0.25063
1.999	0.25006
2.001	0.24994
2.01	0.24938
2.1	0.24390

From the table, the outputs appear to get closer and closer to

$$0.25 = \frac{1}{4}$$

as x gets closer and closer to 2. Therefore, we can say (more precisely, *predict*) that the limit of $f(x) = \dfrac{x-2}{x^2-4}$ as x goes to 2 is $\dfrac{1}{4}$. Symbolically:

$$\lim_{x \to 2} \frac{x-2}{x^2-4} = \frac{1}{4}.$$

Important

Even though the function is undefined at $x = 2$, the limit still exists.

Graphical Interpretation

Example 2.2.

The graph of

$$f(x) = \frac{x-2}{x^2-4}$$

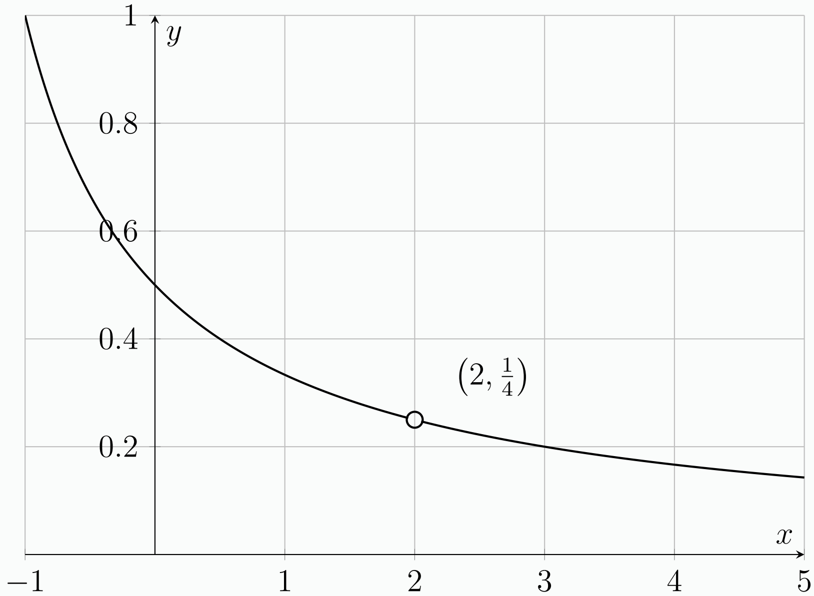
looks like the graph of

$$y = \frac{1}{x+2},$$

except there is a hole at

$$\left(2, \frac{1}{4}\right).$$

Because $f(x)$ is undefined at $x = 2$.



However, the graph of $f(x)$ still approaches the same y-value from both sides as x approaches 2.

2.2. One-Sided Limits

Sometimes we only approach a number from one side.

Left-Hand Limit

$$\lim_{x \to a^-} f(x) = L$$

means that $f(x)$ approaches L as x approaches a from the left (or, from the negative side).

Right-Hand Limit

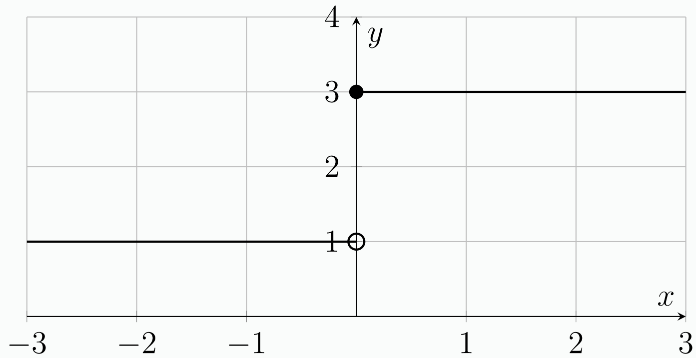
$$\lim_{x \to a^+} f(x) = L$$

means that $f(x)$ approaches L as x approaches a from the right (or, from the positive side).

Example: One-Sided Limits Are Different

Example 2.3.

Consider the function $f(x) = \begin{cases} 1, & x < 0, \\ 3, & x \geq 0. \end{cases}$



$$\lim_{x\to 0^-} f(x) = 1$$

$$\lim_{x\to 0^+} f(x) = 3$$

$$\lim_{x\to 0} f(x) = \text{DNE}$$

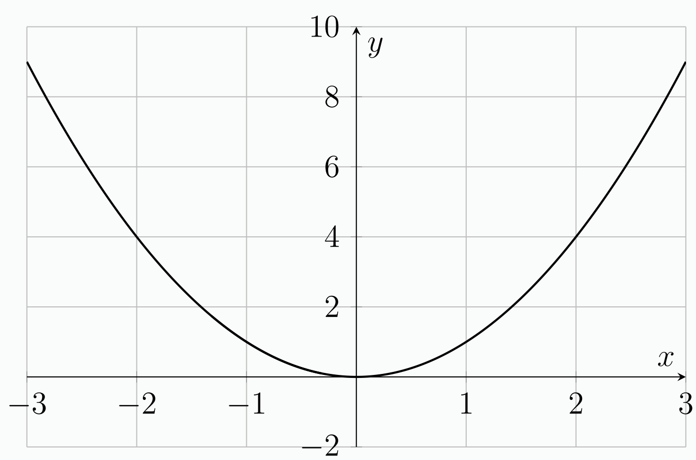
Important

A two-sided limit exists only when the left-hand limit and right-hand limit both exist and are equal.

Example Where the Limit Exists

Example 2.4.

Consider the function $f(x) = x^2$



$$\lim_{x\to 2^-} f(x) = 4$$

$$\lim_{x\to 2^+} f(x) = 4$$

$$\lim_{x\to 2} f(x) = 4$$

2.3. Infinite Limits

Sometimes the values of a function increase or decrease without bound.

Idea of Infinite Limits

If the outputs of a function become arbitrarily large as x approaches a, we write

$$\lim_{x \to a} f(x) = \infty.$$

If the outputs become arbitrarily negative, we write

$$\lim_{x \to a} f(x) = -\infty.$$

Important

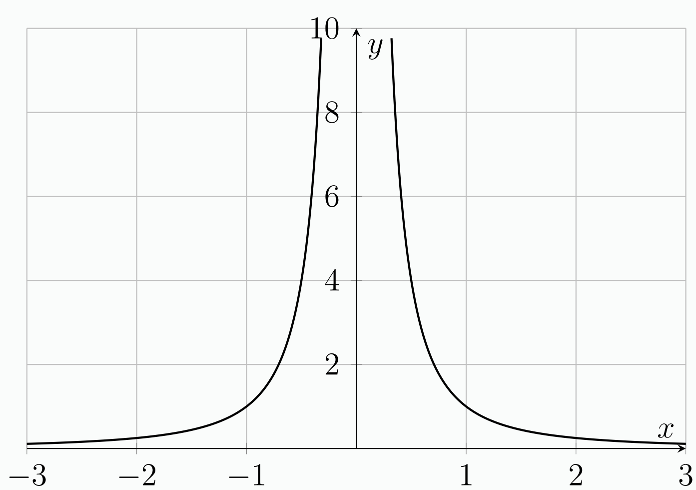
The symbols ∞ and $-\infty$ do not represent real numbers. They describe unbounded behavior.

Example of an Infinite Limit

Example 2.5.

Consider the function

$$f(x) = \frac{1}{x^2}.$$



$$\lim_{x \to 0^-} f(x) = \infty$$

$$\lim_{x \to 0^+} f(x) = \infty$$

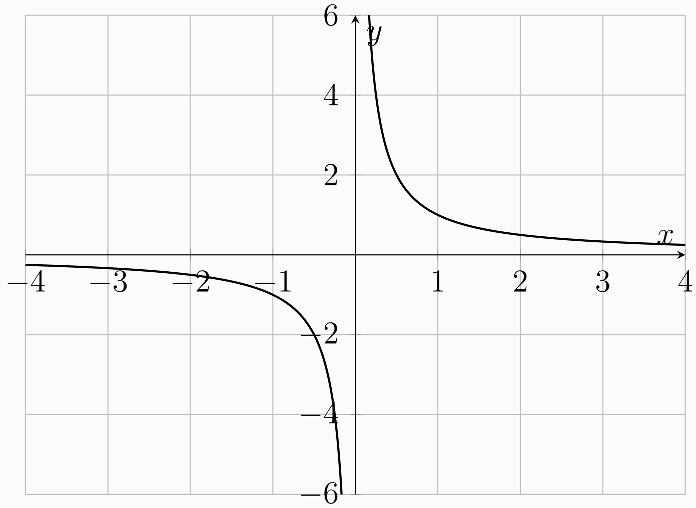
$$\lim_{x \to 0} f(x) = \infty$$

Example with Different Infinite Behavior

Example 2.6.

Consider the function

$$f(x) = \frac{1}{x}.$$



$$\lim_{x \to 0^-} f(x) = -\infty$$

$$\lim_{x \to 0^+} f(x) = \infty$$

$$\lim_{x \to 0} f(x) = \text{DNE}$$

2.4. Summary

- Limits describe the behavior of a function near a point.
- A limit can exist even if the function is undefined at that point.
- One-sided limits study behavior from only one direction.
- A two-sided limit exists only if the two one-sided limits are equal.
- Infinite limits describe functions that grow without bound.

2.5. Calculating Limits Using the Limit Laws

In many situations, limits can be calculated using algebraic operations and known limit laws instead of tables or graphs.

Basic Limit Laws

Suppose that c is a constant and the limits

$$\lim_{x \to a} f(x)$$

and

$$\lim_{x \to a} g(x)$$

exist.

Then the following laws hold.

1. Sum Law

$$\lim_{x \to a}[f(x) + g(x)] = \lim_{x \to a} f(x) + \lim_{x \to a} g(x)$$

2. Difference Law

$$\lim_{x \to a}[f(x) - g(x)] = \lim_{x \to a} f(x) - \lim_{x \to a} g(x)$$

3. Constant Multiple Law

$$\lim_{x \to a}[cf(x)] = c \lim_{x \to a} f(x)$$

4. Product Law

$$\lim_{x \to a}[f(x)g(x)] = \left(\lim_{x \to a} f(x)\right)\left(\lim_{x \to a} g(x)\right)$$

5. Quotient Law

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)}$$

provided that

$$\lim_{x \to a} g(x) \neq 0.$$

Power Law

The Power Law

If n is a positive integer, then

$$\lim_{x\to a}[f(x)]^n = \left(\lim_{x\to a} f(x)\right)^n.$$

Root Law

The Root Law

If n is a positive integer, then

$$\lim_{x\to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x\to a} f(x)}.$$

If n is even, we assume that

$$\lim_{x\to a} f(x) > 0.$$

Direct Substitution Property

The Direct Substitution Property

If f is a polynomial or rational function and a is in the domain of f, then

$$\lim_{x\to a} f(x) = f(a).$$

2.6. Theorems About Limits

Important

Theorem 1

$$\lim_{x\to a} f(x) = L$$

if and only if

$$\lim_{x\to a^-} f(x) = L = \lim_{x\to a^+} f(x).$$

Important

Theorem 2

If

$$f(x) \leq g(x)$$

when x is near a (except possibly at a) and both limits exist, then

$$\lim_{x \to a} f(x) \leq \lim_{x \to a} g(x).$$

Theorem 3: The Squeeze Theorem (The Sandwich Theorem)

If

$$f(x) \leq g(x) \leq h(x)$$

when x is near a (except possibly at a) and

$$\lim_{x \to a} f(x) = \lim_{x \to a} h(x) = L,$$

then

$$\lim_{x \to a} g(x) = L.$$

Polynomial Function

Example 2.7.

Evaluate:

$$\lim_{x \to 2} (3x^2 - 5x + 1)$$

Solution.

Just replace the variable with the limit. If we get a number, then that is the limit. Else we may need to perform some algebraic simplifications first.

$$\begin{aligned} \lim_{x \to 2} (3x^2 - 5x + 1) &= 3(2)^2 - 5(2) + 1 \\ &= 12 - 10 + 1 \\ &= \boxed{3}. \end{aligned}$$

Product Law

Example 2.8.
Evaluate:
$$\lim_{x\to -1}(x^4-3x)(x^2+5x+3)$$

Solution.

$$\begin{aligned}\lim_{x\to -1}(x^4-3x)(x^2+5x+3) &= ((-1)^4-3(-1))\cdot((-1)^2+5(-1)+3)\\ &= (1+3)(1-5+3)\\ &= (4)(-1)\\ &= \boxed{-4}.\end{aligned}$$

Quotient Law

Example 2.9.
Evaluate:
$$\lim_{t\to -2}\frac{t^4-2}{2t^2-3t+2}$$

Solution.

$$\begin{aligned}\lim_{t\to -2}\frac{t^4-2}{2t^2-3t+2} &= \frac{(-2)^2-2}{2(-2)^2-3(-2)+2}=\frac{2}{16}\\ &= \boxed{\frac{1}{8}}.\end{aligned}$$

Root Law

Example 2.10.
Evaluate:
$$\lim_{u\to 2}\sqrt{u^4+3u+6}$$

Solution.

$$\begin{aligned}\lim_{u\to 2}\sqrt{u^4+3u+6} &= \sqrt{2^4+3(2)+6}=\sqrt{28}\\ &= \boxed{2\sqrt{7}}.\end{aligned}$$

Rational Function

Example 2.11.
Evaluate:

$$\lim_{x\to 2}\sqrt{\frac{2x^2+1}{3x-2}}$$

Solution.

$$\begin{aligned}\lim_{x\to 2}\sqrt{\frac{2x^2+1}{3x-2}} &= \sqrt{\frac{2(2)^2+1}{3(2)-2}}\\ &= \sqrt{\frac{9}{8}}\\ &= \boxed{\frac{3}{2\sqrt{2}}}\end{aligned}$$

Factor and Simplify

Example 2.12.
Evaluate:

$$\lim_{x\to -3}\frac{x^2+3x}{x^2-x-12}$$

Solution.

$$\begin{aligned}\lim_{x\to -3}\frac{x^2+3x}{x^2-x-12} &= \lim_{x\to -3}\frac{x(x+3)}{(x-4)(x+3)}\\ &= \lim_{x\to -3}\frac{x}{x-4}\\ &= \frac{-3}{-3-4}\\ &= \boxed{\frac{3}{7}}.\end{aligned}$$

Squeeze Theorem

Example 2.13.
Suppose

$$4x - 9 \leq f(x) \leq x^2 - 4x + 7$$

for $x \geq 0$. Find

$$\lim_{x \to 4} f(x).$$

Solution.
We have that

$$\lim_{x \to 4}(4x - 9) = \lim_{x \to 4}(x^2 - 4x + 7) = 7.$$

Therefore, by the squeeze theorem,

$$\boxed{\lim_{x \to 4} f(x) = 7}$$

One-Sided Limits

Example 2.14.
Let

$$f(x) = \begin{cases} x^2 + 1, & x < 1, \\ (x-2)^2, & x \geq 1. \end{cases}$$

Find:

(a). $\lim_{x \to 1^-} f(x)$ (b). $\lim_{x \to 1^+} f(x)$ (c). $\lim_{x \to 1} f(x)$.

Solution.

(a). $\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} (x^2 + 1) = (1)^2 + 1 = \boxed{2}$.

(b). $\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} (x - 2)^2 = (1 - 2)^2 = \boxed{1}$.

(c). Because $\lim_{x \to 1^-} f(x) \neq \lim_{x \to 1^+} f(x)$, the limit $\lim_{x \to 1} f(x)$ does not exist.

Exercises

1. Evaluate
$$\lim_{x\to 3}(2x^2 - 5x + 4).$$

2. Evaluate
$$\lim_{x\to -2}(x^2 - 4x)(3x + 1).$$

3. Evaluate
$$\lim_{x\to 2}\frac{x^2 + 5}{3x - 1}.$$

4. Evaluate
$$\lim_{x\to 4}\sqrt{x^2 + 9}.$$

5. Evaluate
$$\lim_{x\to 5}\frac{x^2 - 25}{x - 5}.$$

6. Evaluate
$$\lim_{x\to 0}\frac{\sqrt{x + 9} - 3}{x}.$$

7. Find
$$\lim_{x\to 2^-} f(x), \qquad \lim_{x\to 2^+} f(x), \qquad \lim_{x\to 2} f(x),$$
where
$$f(x) = \begin{cases} x + 1, & x < 2, \\ x^2 - 1, & x \geq 2. \end{cases}$$

8. Evaluate
$$\lim_{x\to 0^-}\frac{1}{x}, \qquad \lim_{x\to 0^+}\frac{1}{x}, \qquad \lim_{x\to 0}\frac{1}{x}.$$

9. Suppose

$$2x - 1 \le f(x) \le x^2 - 2x + 5.$$

   Find

$$\lim_{x \to 3} f(x).$$

10. Evaluate

$$\lim_{x \to 0} \frac{\sin(4x)}{x}.$$

11. Evaluate

$$\lim_{x \to 0} \frac{1 - \cos x}{x}.$$

12. Evaluate

$$\lim_{x \to 4} \frac{\sqrt{x} - 2}{x - 4}.$$

Chapter 3

Continuity

3.1. Continuity

Why Is Continuity Important?

Many of the theorems in calculus require continuity as an assumption.
For example, the Intermediate Value Theorem, the Extreme Value Theorem, and the Mean Value Theorem are all true only because the functions involved are continuous.
Whenever you encounter an important theorem in calculus, always ask yourself:

"Is the function continuous?"

It is often the first hypothesis that must be checked.

1. Definition of Continuity at a Point

Continuity

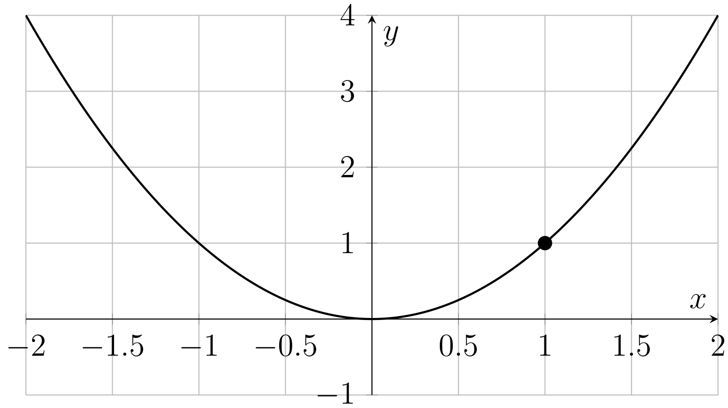
A function f is **continuous at a number** a if

$$\lim_{x \to a} f(x) = f(a).$$

This means the function $f(x)$ is continuous at $x = a$ if:

1. $f(a)$ exists,
2. $\lim_{x \to a} f(x)$ exists,
3. and
$$\lim_{x \to a} f(x) = f(a).$$

Graphical Meaning



The function above is continuous because the graph has no breaks or holes.

3.2. Meaning of Discontinuity

A function is **discontinuous** at a point if it is not continuous there.

Common types of discontinuities include:

1. Hole (removable discontinuity)
2. Jump discontinuity
3. Infinite discontinuity

Example: Hole

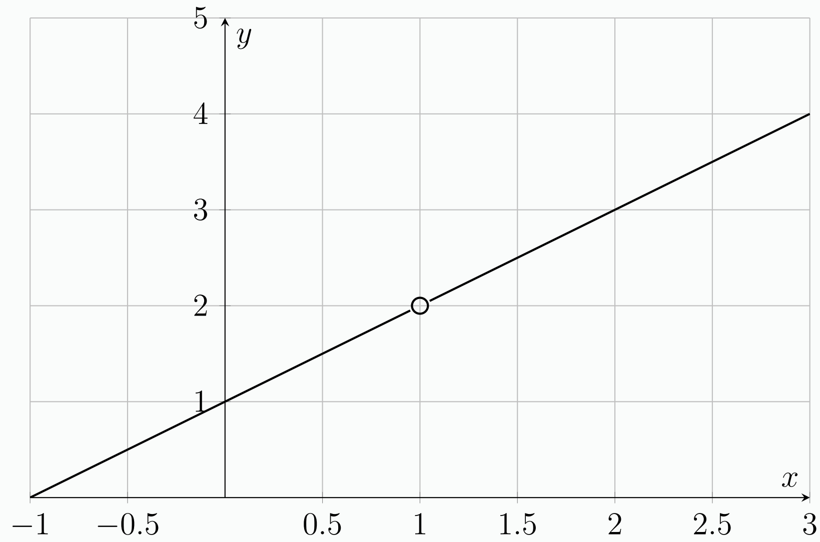
Example 3.1.

$$f(x) = \frac{x^2 - 1}{x - 1}$$

is undefined at $x = 1$. However,

$$f(x) = x + 1 \quad \text{for } x \neq 1.$$

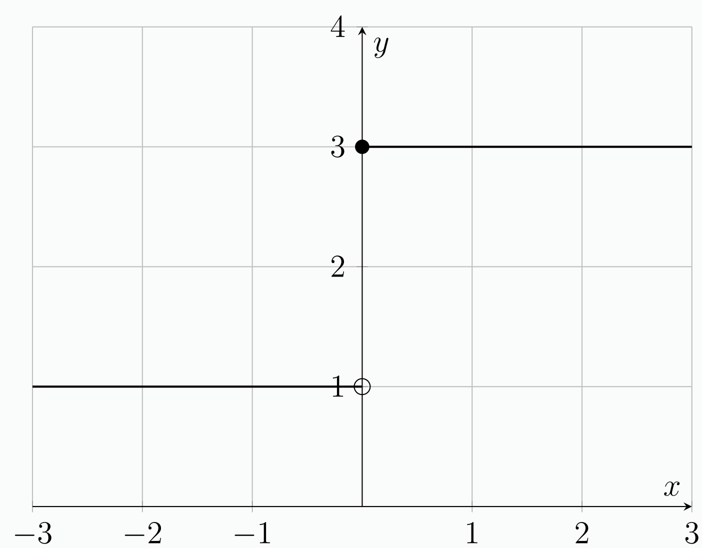
Thus the graph has a hole at $(1, 2)$.



Example: Jump Discontinuity

Example 3.2.

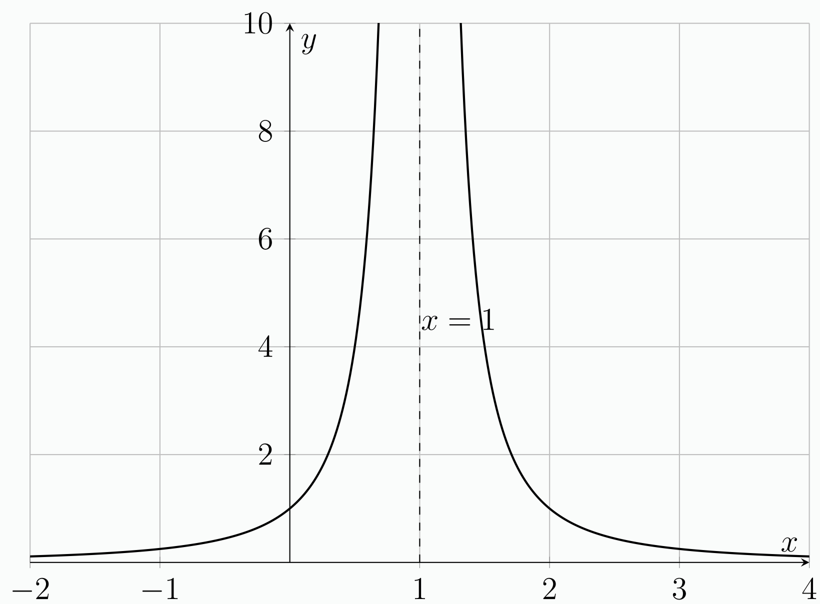
$$f(x) = \begin{cases} 1, & x < 0, \\ 3, & x \geq 0. \end{cases}$$



The function $f(x)$ is continuous on the intervals $(-\infty, 0)$ and $(0, \infty)$. However, when x crosses 0, the function $f(x)$ jumps from $f(x) \equiv 1$ to $f(x) \equiv 3$.

Example: Infinite Discontinuity

Example 3.3.
Consider the function
$$f(x) = \frac{1}{(x-1)^2}.$$
As x approaches 1 from either side,
$$(x-1)^2$$
gets very close to 0, so the function values become extremely large.
Thus,
$$\lim_{x\to 1} \frac{1}{(x-1)^2} = \infty.$$
The graph has a vertical asymptote at
$$x = 1.$$
Therefore, the function has an **infinite discontinuity** at $x = 1$.



Important

Infinite discontinuities occur when the function increases or decreases without bound near a point. These are usually associated with vertical asymptotes.

3.3. Right Continuity and Left Continuity

Right Continuity

A function f is **continuous from the right** at a if

$$\lim_{x \to a^+} f(x) = f(a).$$

Left Continuity

A function f is **continuous from the left** at a if

$$\lim_{x \to a^-} f(x) = f(a).$$

3.4. Continuity on an Interval

Continuity on an Interval

A function is continuous on an open interval (a, b) if it is continuous at every point in the interval.

Continuity on Closed Intervals

Important

A function is continuous on the closed interval $[a, b]$ if

1. it is continuous on (a, b),
2. it is continuous from the right at a,
3. it is continuous from the left at b.

3.5. Continuity Properties

Important

If f and g are continuous at a and c is a constant, then the following functions are also continuous at a:

1. $f + g$
2. $f - g$
3. cf
4. fg
5. $\dfrac{f}{g}$ if $g(a) \neq 0$

Composition of Continuous Functions

Important

If g is continuous at a and f is continuous at $g(a)$, then

$$f \circ g$$

is continuous at a. Recall that $(f \circ g)(x) = f(g(x))$.

Equivalently,

$$\lim_{x \to a} f(g(x)) = f\left(\lim_{x \to a} g(x)\right).$$

3.6. Some Important Continuous Functions

Important

The following types of functions are continuous at every point (number) in their domains:

- polynomials
- rational functions
- root functions
- exponential functions
- logarithmic functions
- trigonometric functions

3.7. Limit Through a Continuous Function

Important

If f is continuous at b and

$$\lim_{x\to a} g(x) = b,$$

then

$$\lim_{x\to a} f(g(x)) = f(b).$$

In other words,

$$\lim_{x\to a} f(g(x)) = f\left(\lim_{x\to a} g(x)\right).$$

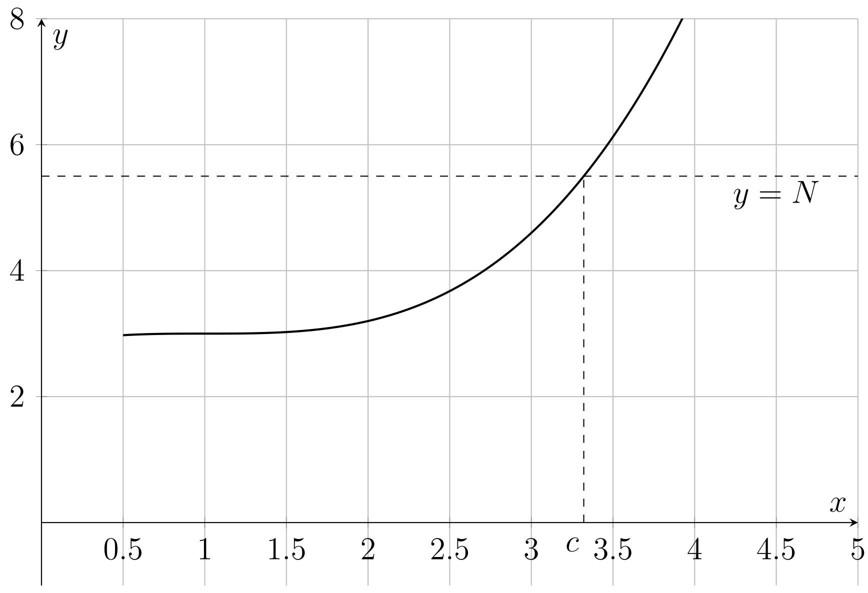
3.8. The Intermediate Value Theorem

The Intermediate Value Theorem

Suppose that f is continuous on $[a, b]$. Let N be any number between $f(a)$ and $f(b)$. Then there exists a number c in (a, b) such that

$$f(c) = N.$$

Graphical Meaning



The graph must cross the horizontal line $y = N$ somewhere. That is, there is c such that $f(c) = N$.

Piecewise Function

Example 3.4.
Determine whether the function is continuous at $x = 1$.

$$f(x) = \begin{cases} 1 - x^2, & x < 1, \\ 1/x, & x \geq 1. \end{cases}$$

Solution.
We have

$$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} (1 - x^2) = 1 - (1)^2 = 0,$$

and

$$\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} \frac{1}{x} = \infty.$$

So,

$$\lim_{x \to 1^-} f(x) \neq \lim_{x \to 1^+} f(x).$$

Thus, $\lim\limits_{x \to 1} f(x)$ does not exists and therefore $f(x)$ is not continuous at $x = 1$.

Removable Discontinuity

Example 3.5.
Determine whether the function is continuous at $x = 1$.

$$f(x) = \begin{cases} \dfrac{x^2 - x}{x^2 - 1}, & x \neq 1, \\ 1, & x = 1. \end{cases}$$

Solution.
We have

$$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} \frac{x^2 - x}{x^2 - 1} = \lim_{x \to 1^-} \frac{x(x-1)}{(x-1)(x+1)} = \lim_{x \to 1^-} \frac{x}{(x+1)} = \frac{1}{2},$$

and

$$\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} \frac{x^2 - x}{x^2 - 1} = \lim_{x \to 1^+} \frac{x(x-1)}{(x-1)(x+1)} = \lim_{x \to 1^+} \frac{x}{(x+1)} = \frac{1}{2},$$

So,

$$\lim_{x\to 1} f(x) = \lim_{x\to 1^-} f(x) = \lim_{x\to 1^+} f(x) = \frac{1}{2}.$$

However, $f(1) = 1$.
Thus,

$$\lim_{x\to 1} f(x) \neq f(1),$$

and therefore, $f(x)$ is not continuous at $x = 1$.

Example 3.6.

Determine whether the function is continuous at $x = 0$.

$$f(x) = \begin{cases} \cos x, & x < 0, \\ 0, & x = 0, \\ 1 - x^2, & x > 0. \end{cases}$$

Solution.
Notice that

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} \cos x = \cos(0) = 1,$$

and

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^-} (1 - x^2) = 1 - 0^2 = 1.$$

Thus,

$$\lim_{x\to 0} f(x) = 1.$$

However, $f(0) = 0$. Hence, $f(x)$ is not continuous at $x = 0$.

Continuity on the Domain

Example 3.7.

Determine where the function is continuous.

$$B(x) = \frac{\tan x}{\sqrt{4 - x^2}}$$

Solution.
Notice that the domain of $B(x)$ is

$$\boxed{\{x : 4 - x^2 \geq 0\} \setminus \left\{\pm\frac{\pi}{2}\right\} = [-4, 4] \setminus \left\{\pm\frac{\pi}{2}\right\}}$$

Because $\tan x$ and $\sqrt{4 - x^2}$ are continuous on this set, $B(x)$ is also continuous there.

Example 3.8.

Determine where the function is continuous.

$$M(x) = \sqrt{1 + \frac{1}{x}}$$

Solution.

The domain of $M(x)$ is

$$\boxed{\left\{x : 1 + \frac{1}{x} \geq 0 \text{ and } x \neq 0\right\} = (-\infty, -1] \cup (0, \infty)}$$

The function $M(x)$ is continuous on this domain.

Example 3.9.

Evaluate:

$$\lim_{x \to \pi} \sin(x + \sin x)$$

Solution.

$$\begin{aligned}
\lim_{x \to \pi} \sin(x + \sin x) &= \sin(\pi + \sin \pi) \\
&= \sin(\pi + 0) \\
&= \sin \pi \\
&= 0.
\end{aligned}$$

$$\boxed{\lim_{x \to \pi} \sin(x + \sin x) = 0}$$

Example 3.10.

Evaluate:

$$\lim_{x \to \frac{\pi}{4}} x^2 \tan x$$

Solution.

$$\begin{aligned}\lim_{x\to\frac{\pi}{4}} x^2\tan x &= \left(\frac{\pi}{4}\right)^2\tan\left(\frac{\pi}{4}\right)\\ &= \left(\frac{\pi}{4}\right)^2\cdot 1\\ &= \left(\frac{\pi}{4}\right)^2\\ &= \frac{\pi^2}{16}.\end{aligned}$$

$$\boxed{\lim_{x\to\frac{\pi}{4}} x^2\tan x = \frac{\pi^2}{16}}$$

Application of Continuity

Example 3.11.

Suppose f and g are continuous functions such that

$$g(2)=6$$

and

$$\lim_{x\to 2}[3f(x)+f(x)g(x)]=36.$$

Find $f(2)$.

Solution.

Because f and g are continuous, we get

$$\begin{aligned}\lim_{x\to 2}[3f(x)+f(x)g(x)] &= 36\\ 3f(2)+f(2)g(2) &= 36\\ 3f(2)+6f(2) &= 36\\ 8f(2) &= 36\\ f(2) &= \frac{36}{8}\\ f(2) &= \frac{9}{2}.\end{aligned}$$

$$\boxed{f(2)=\frac{9}{2}}$$

Intermediate Value Theorem

Example 3.12.
Show that the equation

$$x^3 = x + 1$$

has a solution.
Solution.
Let

$$f(x) := x^3 - x - 1.$$

Then, $f(x)$ is a polynomial function and therefore continuous on $\mathbb{R}$ (the set of all real numbers). Notice that

$$f(-1) = (-1)^3 - (-1) - 1 = -1 < 0$$

and

$$f(2) = 2^3 - (2) - 1 = 5 > 0.$$

Therefore by Intermediate Value Theorem, there is a number c in $[-1, 2]$ such that $f(c) = 0$. That is, there is a number c in $[-1, 2]$ such that

$$c^3 = c + 1.$$

Example 3.13.
Suppose f is continuous on $[1, 5]$ and the only solutions of

$$f(x) = 6$$

are

$$x = 1 \quad \text{and} \quad x = 4.$$

If

$$f(2) = 8$$

, explain why

$$f(3) > 6.$$

Solution.
Let

$$g(x) := f(x) - 6.$$

Then,

$$g(1) = g(4) = 0$$

and

$$g(2) = f(2) - 6 = 8 - 6 = 2 > 0.$$

Because $g(x)$ is continuous with $g(2) > 0$, and $x = 1$ and $x = 4$ are the only solutions to the equation $g(x) = 0$, we must have

$$g(3) > 0.$$

In fact, for any a in $(1, 4)$, we must have $g(a) > 0$. This implies

$$f(3) - 6 > 0$$

. Thus,

$$f(3) > 6.$$

Exercises

1. Determine whether
$$f(x) = \frac{x^2 - 4}{x - 2}$$
is continuous at $x = 2$.

2. Determine whether
$$f(x) = |x|$$
is continuous at $x = 0$.

3. Find where
$$f(x) = \sqrt{9 - x^2}$$
is continuous.

4. Evaluate: $\lim_{x \to 0} \cos(x^2)$.

5. Use the Intermediate Value Theorem to explain why
$$x^3 - 2x - 5 = 0$$
has a solution between 2 and 3.

6. For what values of x is the function continuous?
$$f(x) = \begin{cases} 0, & \text{if } x \text{ is rational,} \\ 1, & \text{if } x \text{ is irrational.} \end{cases}$$

7. For what values of x is the function continuous?
$$g(x) = \begin{cases} 0, & \text{if } x \text{ is rational,} \\ x, & \text{if } x \text{ is irrational.} \end{cases}$$

8. Is there a number that is exactly 1 more than its cube?

9. Show that the function
$$f(x) = \begin{cases} x^4 \sin(1/x), & \text{if } x \neq 0, \\ 0, & \text{if } x = 0 \end{cases}$$
is continuous on $(-\infty, \infty)$.

10. A Tibetan monk leaves the monastery at 6:00 AM and takes his usual path to the top of the mountain, arriving at 6:00 PM. The following morning, he starts at 6:00 AM at the top and takes the same path back, arriving at the monastery at 6:00 PM.

    Use the Intermediate Value Theorem to show that there is a point on the path where the monk crosses at exactly the same time of day on both days.

Chapter 4

Derivatives and Rates of Change

4.1. Derivatives and Rates of Change

Why Should We Care About Derivatives?

The derivative measures how a quantity changes.
Engineers use derivatives to study motion. Economists use derivatives to analyze marginal cost and marginal revenue. Physicists use derivatives to describe velocity and acceleration. Biologists use derivatives to model population growth.
Throughout this book, the derivative will become one of the most powerful tools for understanding change.

4.2. Tangent Lines

Tangent Line

The **tangent line** to the curve

$$y = f(x)$$

at the point

$$P(a, f(a))$$

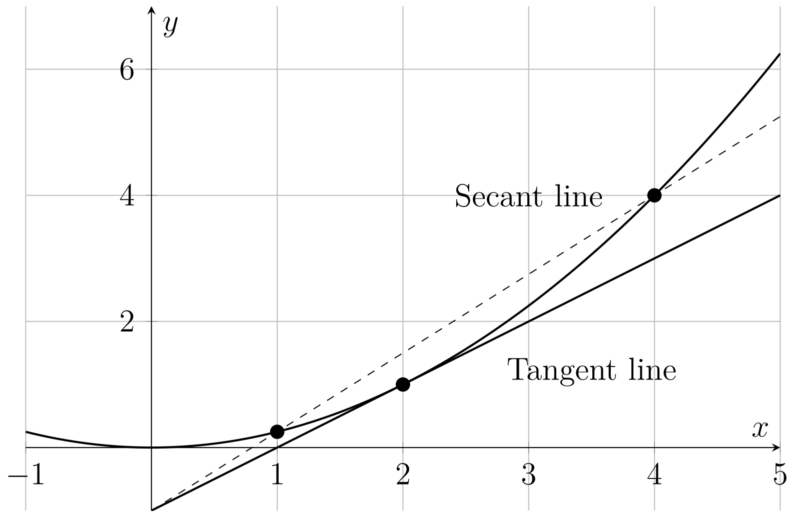
is the line through P with slope

$$m = \lim_{x \to a} \frac{f(x) - f(a)}{x - a},$$

provided that this limit exists.

4.2.1 Secant Line vs Tangent Line

A secant line passes through two points on the curve. A tangent line touches the curve at one point and has the same instantaneous slope as the curve there.



4.3. Alternative Definition of Tangent Slope

Sometimes the tangent slope is written using h instead of $x - a$.

Slope

The slope of the tangent line can also be written as

$$m = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}.$$

Both formulas are equivalent. The reason is that if $h = x - a$, then

$$\boldsymbol{x \to a} \text{ is equivalent to } \boldsymbol{h \to 0}.$$

4.4. Velocity from a Position Function

4.4.1 Average Velocity

Suppose the position of an object at time t is given by a function $s(t)$. Then, the average velocity over the interval $[t_1, t_2]$ is

$$\frac{s(t_2) - s(t_1)}{t_2 - t_1}.$$

This is the slope of the secant line.

4.4.2 Instantaneous Velocity

The instantaneous velocity at time $t = a$ is

$$v(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}.$$

This is the slope of the tangent line.

Average Velocity Example

Example 4.1.
Suppose the position function is

$$s(t) = t^2 + 1.$$

Find the average velocity on the interval $[1, 3]$.
Solution.

$$Avg(v) = \frac{s(3) - s(1)}{3 - 1} = \frac{10 - 2}{2} = 4.$$

Instantaneous Velocity Example

Example 4.2.
Suppose

$$s(t) = t^2 + 1.$$

Find the instantaneous velocity at $t = 2$.
Solution.
The instantaneous velocity at $t = 2$ is

$$\begin{aligned}
\lim_{h \to 0} \frac{s(h+2) - s(2)}{h} &= \lim_{h \to 0} \frac{((h+2)^2 + 1) - (2^2 + 1)}{h} \\
&= \lim_{h \to 0} \frac{h^2 + 4h + 5 - 5}{h} \\
&= \lim_{h \to 0} \frac{h^2 + 4h}{h} \\
&= \lim_{h \to 0} (h + 4) \\
&= \boxed{4}.
\end{aligned}$$

4.5. Definition of the Derivative

Derivative

The **derivative** of a function f at a number a, denoted by

$$f'(a),$$

is

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h},$$

if this limit exists.

Important

$f'(a)$ is read as "f prime of a."

Meaning of the Derivative

The derivative represents:

- slope of the tangent line,
- instantaneous rate of change,
- instantaneous velocity.

4.6. Rates of Change

Average Rate of Change

Average Rate of Change

The difference quotient

$$\frac{\Delta y}{\Delta x} = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

is called the **average rate of change** of y with respect to x over the interval $[x_1, x_2]$.

Geometric Meaning

- The average rate of change is the slope of the secant line.
- The instantaneous rate of change is the slope of the tangent line.

4.6.1 Slope of a Tangent Line

Example 4.3.
Find the slope of the tangent line to the curve

$$y = x - x^3$$

at the point $(1, 0)$.

Solution.
Let $y = f(x)$. Then, the slope of the tangent line to the curve $f(x)$ at the point $(0, 1)$ is:

$$\begin{aligned}
\lim_{h\to 0} \frac{f(1+h) - f(1)}{h} &= \lim_{h\to 0} \frac{[(1+h) - (1+h)^3] - (0)}{h} \\
&= \lim_{h\to 0} \frac{(1+h)[1 - (1+h)^2]}{h} \\
&= \lim_{h\to 0} \frac{(1+h)[1 - (1+h)][1 + (1+h)]}{h} \\
&= \lim_{h\to 0} \frac{(1+h)(-h)(2+h)}{h} \\
&= \lim_{h\to 0} (1+h)(-1)(2+h) \\
&= \boxed{-2}.
\end{aligned}$$

4.6.2 Tangent Line Equation

Example 4.4.
Find an equation of the tangent line to

$$y = \sqrt{x}$$

at the point $(1, 1)$.

Solution.
The slope of the tangent line is

$$\lim_{h\to 0}\frac{\sqrt{1+h}-\sqrt{1}}{h} = \lim_{h\to 0}\frac{\sqrt{1+h}-1}{h}$$
$$= \lim_{h\to 0}\frac{(\sqrt{1+h}-1)}{h}\cdot\frac{\sqrt{1+h}+1}{\sqrt{1+h}+1}$$

$$= \lim_{h\to 0}\frac{1+h-1}{h(\sqrt{1+h}+1)}$$
$$= \lim_{h\to 0}\frac{h}{h(\sqrt{1+h}+1)}$$
$$= \lim_{h\to 0}\frac{1}{\sqrt{1+h}+1}$$
$$= \boxed{\frac{1}{2}}.$$

Thus, the equation of the tangent line at the point $(1,1)$ is

$$\boxed{y-1=\frac{1}{2}(x-1)}$$

Equivalently,

$$\boxed{y=\frac{1}{2}x+\frac{1}{2}}$$

Exercises

1. Find the average rate of change of

$$f(x) = x^2 + 1$$

on the interval $[1, 4]$.

2. Find the slope of the tangent line to $y = x^2$ at $x = 1$ using the definition of derivative.

3. Find the instantaneous velocity if

$$s(t) = t^2 - 4t + 1$$

at $t = 3$.

4. Find an equation of the tangent line to

$$y = x^2 - 3x + 2$$

at the point $(1, 0)$.

5. Find the derivative of $f(x) = \sqrt{x}$ at $x = 4$ using the definition.

Chapter 5

The Derivative as a Function

5.1. The Derivative as a Function

In this section, we study the derivative as a function. In the previous section, we defined the derivative at a fixed number a:

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}.$$

Now we allow the number a to vary. Replacing a by x, we get

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$

Important

The derivative $f'(x)$ is itself a function. It gives the slope of the tangent line to the graph of f at each value of x where the derivative exists.

Derivative Notation

If

$$y = f(x),$$

then the derivative can be written in several ways:

$$f'(x) = y' = \frac{dy}{dx} = \frac{d}{dx} f(x) = Df(x) = D_x f(x).$$

Important

The symbols $\dfrac{d}{dx}$ and D are called differentiation operators.

5.1.1 Differentiability

Differentiable Function

A function f is differentiable at a if $f'(a)$ exists.
A function is differentiable on an interval if it is differentiable at every number in that interval.

Important

If f is differentiable at a, then f is continuous at a.

The converse is not always true. A function may be continuous at a point but not differentiable there.

How Can a Function Fail to Be Differentiable?

A function can fail to be differentiable at a point for several reasons.

1. The graph has a corner.
2. The function is discontinuous.
3. The graph has a vertical tangent line.

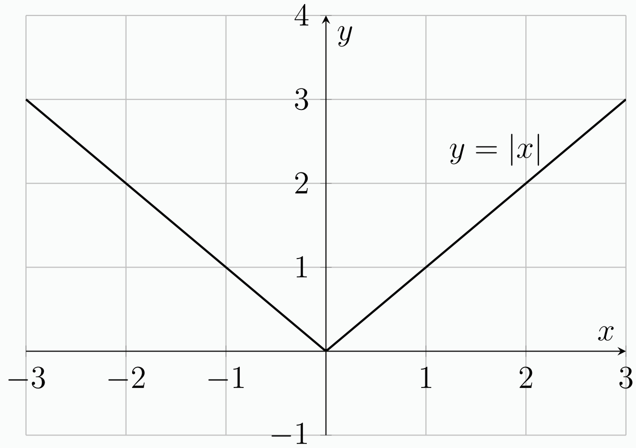
Example: Corner

Example 5.1.
Consider $f(x) = |x|$.
This function is continuous at $x = 0$, but it is not differentiable at $x = 0$.

For $x < 0$, the slope is -1.

For $x > 0$, the slope is 1.

Since the left-hand and right-hand slopes are different, $f'(0)$ does not exist.



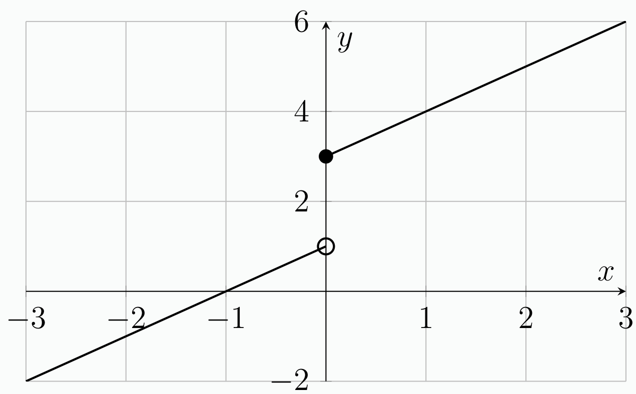
Example: Discontinuity

Example 5.2.

Consider

$$f(x) = \begin{cases} x+1, & x < 0, \\ x+3, & x \geq 0. \end{cases}$$

This function has a jump discontinuity at $x = 0$.
Therefore, f is not differentiable at $x = 0$.



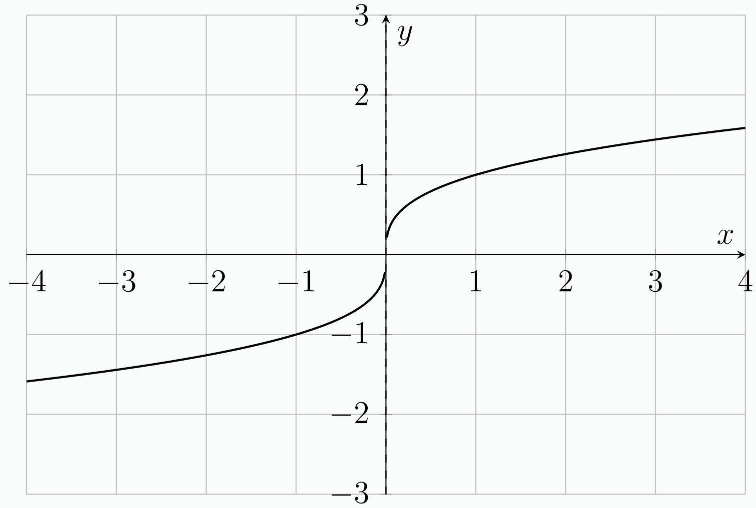
Example: Vertical Tangent

Example 5.3.

Consider $f(x) = \sqrt[3]{x}$.

This function is continuous at $x = 0$, but it has a vertical tangent line at $x = 0$.

Thus $f'(0)$ does not exist.



5.1.2 Higher Derivatives

If f is differentiable, then f' may also be differentiable. The derivative of f' is called the second derivative.

$$(f')' = f''.$$

In Leibniz notation,

$$\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d^2y}{dx^2}.$$

Important

The second derivative is the derivative of the first derivative.

Similarly, the n-th derivative of $y = f(x)$ with respect to x, if it exists, is denoted by

$$\frac{d^nf}{dx^n} \text{ or } \frac{d^ny}{dx^n}.$$

5.2. Finding Derivatives Using the Definition

Example 5.4.

Find the derivative of the function using the definition of derivative. State the domain of the function and the domain of its derivative.

$$f(x) = 4 + 8x.$$

Solution.

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h\to 0} \frac{[4 + 8(x+h)] - (4 + 8x)}{h} \\ &= \lim_{h\to 0} \frac{8h}{h} \\ &= 8. \end{aligned}$$

$$\boxed{\begin{array}{l} f'(x) = 8 \\ \mathrm{Dom}(f) = \mathrm{Dom}(f') = \mathbb{R} \end{array}}$$

Example 5.5.

Find the derivative of the function using the definition of derivative. State the domain of the function and the domain of its derivative.

$$f(x) = x^2 - 2x^3.$$

Solution.

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h\to 0} \frac{[(x+h)^2 - 2(x+h)^3] - (x^2 - 2x^3)}{h} \\ &= \lim_{h\to 0} \frac{(x+h)^2 - x^2 + 2x^3 - 2(x+h)^3}{h} \\ &= \lim_{h\to 0} \frac{h(2x+h) + 2(-h)[x^2 + x(x+h) + (x+h)^2]}{h} \\ &= \lim_{h\to 0} (2x + h - 2[x^2 + x(x+h) + (x+h)^2]) \end{aligned}$$

$$
\begin{aligned}
&= 2x - 2(x^2 + x^2 + x^2)\\
&= 2x - 6x^3.
\end{aligned}
$$

$$\boxed{\begin{array}{l} f'(x) = 2x - 6x^3 \\ \mathrm{Dom}(f) = \mathrm{Dom}(f') = \mathbb{R} \end{array}}$$

Example 5.6.

Find the derivative of the function using the definition of derivative. State the domain of the function and the domain of its derivative.

$$f(x) = \frac{1}{x}.$$

Solution.

$$
\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}\\
&= \lim_{h\to 0} \frac{\frac{1}{x+h} - \frac{1}{x}}{h}\\
&= \lim_{h\to 0} \frac{\frac{x-(x+h)}{x(x+h)}}{h}\\
&= \lim_{h\to 0} \frac{-h}{hx(x+h)}\\
&= \lim_{h\to 0} \frac{-1}{x(x+h)}\\
&= -\frac{1}{x^2}.
\end{aligned}
$$

$$\boxed{\begin{array}{l} f'(x) = -\dfrac{1}{x^2} \\ \mathrm{Dom}(f) = \mathrm{Dom}(f') = \mathbb{R} \setminus \{0\} = (-\infty, 0) \cup (0, \infty) \end{array}}$$

Example 5.7.

Find the derivative of the function using the definition of derivative. State the domain of the function and the domain of its derivative.

$$f(x) = \sqrt{x}.$$

Solution.

$$
\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \\
&= \lim_{h\to 0} \frac{\sqrt{x+h}-\sqrt{x}}{h} \\
&= \lim_{h\to 0} \frac{\sqrt{x+h}-\sqrt{x}}{h} \cdot \frac{\sqrt{x+h}+\sqrt{x}}{\sqrt{x+h}+\sqrt{x}} \\
&= \lim_{h\to 0} \frac{(x+h)-x}{h(\sqrt{x+h}+\sqrt{x})} \\
&= \lim_{h\to 0} \frac{h}{h(\sqrt{x+h}+\sqrt{x})} \\
&= \lim_{h\to 0} \frac{1}{(\sqrt{x+h}+\sqrt{x})} \\
&= \frac{1}{2\sqrt{x}}
\end{aligned}
$$

$$
\boxed{\begin{array}{l} f'(x) = \dfrac{1}{2\sqrt{x}} \\ \mathrm{Dom}(f) = [0,\infty) \\ \mathrm{Dom}(f') = \mathbb{R}\setminus\{0\} \end{array}}
$$

5.3. Where Is the Function Not Differentiable?

Example 5.8.

Determine where the function

$$g(x) = x + |x|$$

is differentiable. Then find a formula for $g'(x)$.

Solution.

The function $f(x) := |x|$ is not differentiable at $x = 0$. The reason is

$$\lim_{h\to 0^-} \frac{f(0+h)-f(0)}{h} = \lim_{h\to 0^-} \frac{-h}{h} = -1,$$

and

$$\lim_{h\to 0^+} \frac{f(0+h)-f(0)}{h} = \lim_{h\to 0^+} \frac{h}{h} = 1$$

imply that

$$\lim_{h\to 0}\frac{f(0+h)-f(0)}{h}$$

does not exist. That is, $f'(0)$ does not exist.
Because $h(x)=x$ is a polynomial function, it is differentiable everywhere. Therefore, $g(x)$ is differentiable everywhere except at $x=0$.

Example 5.9.
Let

$$g(x)=x^{2/3}.$$

(a) Show that $g'(0)$ does not exist.

(b) If $a\neq 0$, find $g'(a)$.

Solution.

(a) By definition,

$$\begin{aligned} g'(0) &= \lim_{h\to 0}\frac{g(0+h)-g(0)}{h}\\ &= \lim_{h\to 0}\frac{h^{2/3}-0}{h}\\ &= \lim_{h\to 0}\frac{1}{h^{1/3}} \end{aligned}$$

But this limit does not exist as

$$\lim_{h\to 0^-}\frac{1}{h^{1/3}}=-\infty \qquad \text{and} \qquad \lim_{h\to 0^+}\frac{1}{h^{1/3}}=\infty.$$

(b). Let $a\neq 0$. Then,

$$\begin{aligned} g'(a) &= \lim_{h\to 0}\frac{g(a+h)-g(a)}{h}\\ &= \lim_{h\to 0}\frac{(a+h)^{2/3}-a^{2/3}}{h}\\ &= \lim_{h\to 0}\left[\frac{(a+h)^{2/3}-a^{2/3}}{h}\cdot\frac{(a+h)^{4/3}+(a+h)^{2/3}a^{2/3}+a^{4/3}}{(a+h)^{4/3}+(a+h)^{2/3}a^{2/3}+a^{4/3}}\right]\\ &= \lim_{h\to 0}\frac{[(a+h)^{2/3}]^3-(a^{2/3})^3}{h[(a+h)^{4/3}+(a+h)^{2/3}a^{2/3}+a^{4/3}]} \end{aligned}$$

$$
\begin{aligned}
&= \lim_{h\to 0} \frac{(a+h)^2 - a^2}{h[(a+h)^{4/3} + (a+h)^{2/3}a^{2/3} + a^{4/3}]} \\
&= \lim_{h\to 0} \frac{h(2a+h)}{h[(a+h)^{4/3} + (a+h)^{2/3}a^{2/3} + a^{4/3}]} \\
&= \lim_{h\to 0} \frac{2a+h}{[(a+h)^{4/3} + (a+h)^{2/3}a^{2/3} + a^{4/3}]} \\
&= \frac{2a}{a^{4/3} + a^{2/3}a^{2/3} + a^{4/3}} \\
&= \frac{2a}{3a^{4/3}} \\
&= \frac{2}{3}a^{-1/3}.
\end{aligned}
$$

This limit exists because $a \neq 0$. Therefore,

$$\boxed{g'(a) = \frac{2}{3}a^{-\frac{1}{3}}}$$

Example 5.10.

Find the first and second derivatives of

$$f(x) = 2x^2 - 3x + 1.$$

Solution.

$$
\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\
&= \lim_{h\to 0} \frac{[2(x+h)^2 - 3(x+h) + 1] - (2x^2 - 3x + 1)}{h} \\
&= \lim_{h\to 0} \frac{2[(x+h)^2 - x^2] - 3h}{h} \\
&= \lim_{h\to 0} \frac{2h(2x+h) - 3h}{h} \\
&= \lim_{h\to 0} (2(2x+h) - 3) \\
&= 4x - 3.
\end{aligned}
$$

Thus,

$$f'(x) = 4x - 3.$$

To find the second derivative:

$$\begin{aligned}
f''(x) &= \lim_{h\to 0} \frac{f'(x+h) - f'(x)}{h} \\
&= \lim_{h\to 0} \frac{[4(x+h) - 3] - (4x - 3)}{h} \\
&= \lim_{h\to 0} \frac{4h}{h} \\
&= \lim_{h\to 0} 4 \\
&= 4.
\end{aligned}$$

Thus,

$$f''(x) = 4.$$

So,

$$\boxed{\begin{aligned} &f'(x) = 4x - 3 \\ &f''(x) = 4 \end{aligned}}$$

Exercises

1. Find the derivative of the function using the definition of derivative. State the domain of the function and the domain of its derivative.

$$f(x) = 3 + 6x - 4x^2.$$

2. Find the derivative of the function using the definition of derivative. State the domain of the function and the domain of its derivative.

$$f(x) = x^2 - 3x^3.$$

3. Find the derivative of the function using the definition of derivative. State the domain of the function and the domain of its derivative.

$$H(t) = \frac{2-t}{4+t}.$$

4. Find the derivative of the function using the definition of derivative. State the domain of the function and the domain of its derivative.

$$f(x) = \frac{1}{x+2}.$$

5. Find the derivative of the function using the definition of derivative. State the domain of the function and the domain of its derivative.

$$f(x) = \sqrt{x+1}.$$

6. Determine where the function is differentiable:

$$f(x) = |x-3|.$$

7. Determine where the function is differentiable:

$$f(x) = x^{2/3}.$$

8. Sketch the graph of

$$g(x) = x + |x|.$$

Then answer the following:

(a) For what values of x is g differentiable?

(b) Find a formula for g'.

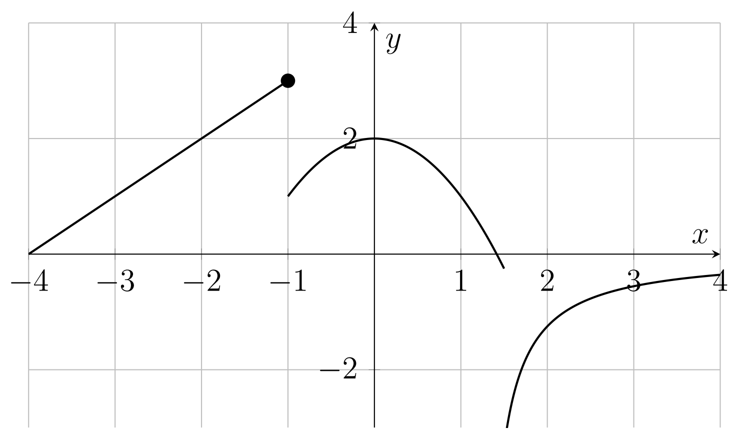
9. Find the first and second derivatives of

$$f(x) = x^4 - 5x^2 + 2x - 1.$$

10. Find the first three derivatives of

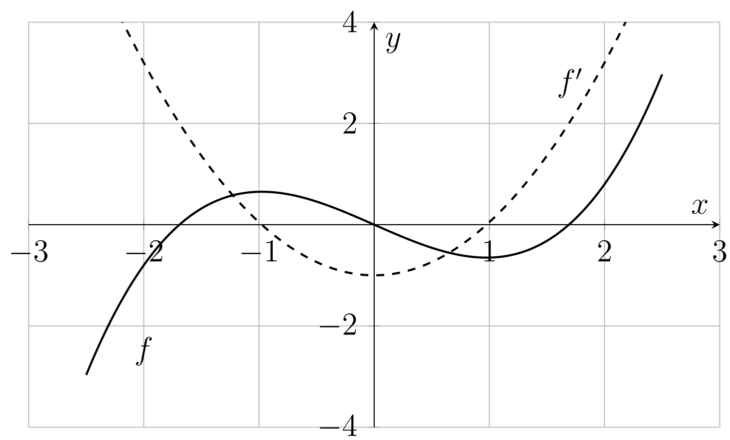
$$f(x) = x^5 - 4x^3 + x.$$

11. State the points where the following graph is not differentiable and explain why.

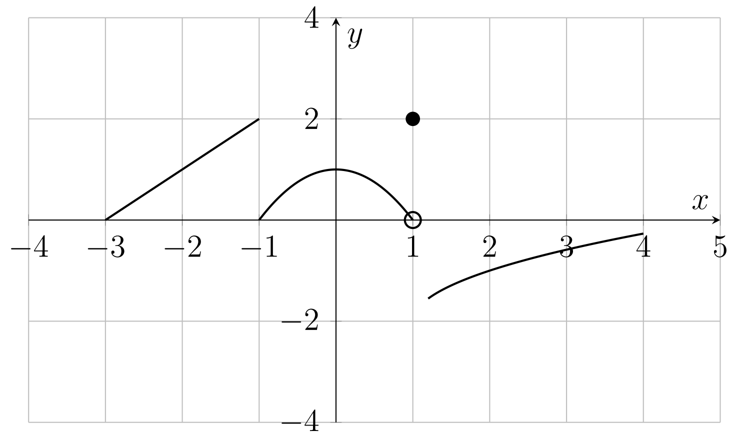


12. The graphs of a function f and its derivative f' are shown. Which is bigger,

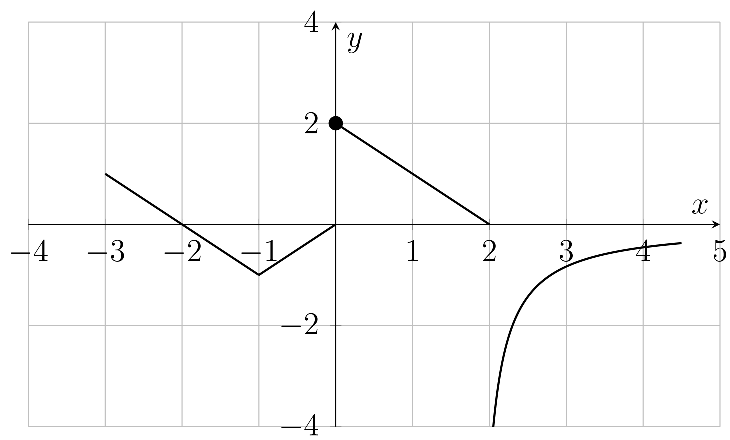
$$f'(-1) \quad \text{or} \quad f''(1)?$$



13. The graph of f is given. State, with reasons, the numbers at which f is not differentiable.

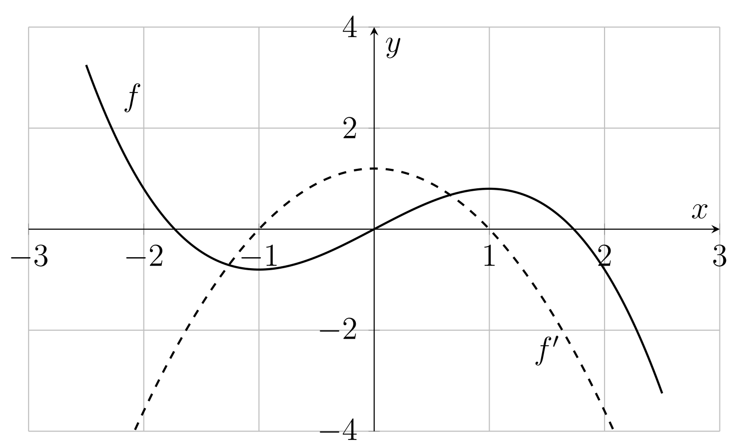


14. The graph of f is given. State, with reasons, the numbers at which f is not differentiable.



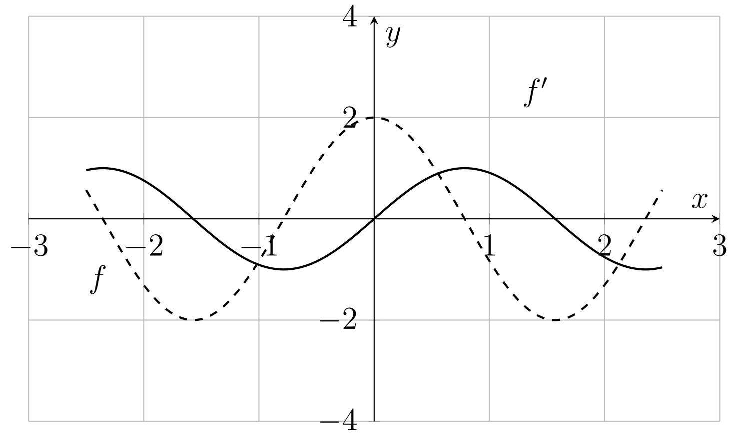
15. The graphs of a function f and its derivative f' are shown. Which is bigger,

$$f'(-1) \quad \text{or} \quad f''(1)?$$



16. The graphs of a function f and its derivative f' are shown. Which is bigger,

$$f'(-1) \quad \text{or} \quad f''(0)?$$



Chapter 6

Differentiation Formulas

Differentiation Formulas

In this section, we study several important differentiation formulas that allow us to quickly compute derivatives. One can think of these as 'tools' for differentiating some basic functions.

A Tip

Whenever you differentiate a complicated expression, do not immediately decide which rule to use. Instead, ask yourself:

1. Is this a sum?
2. Is this a product?
3. Is this a quotient?
4. Is this a composition of functions?

Choosing the correct rule is often more important than carrying out the differentiation.

Derivative of a Constant Function

Important

If c is a constant, then

$$\frac{d}{dx}(c) = 0$$

Example 6.1.

Differentiate the function $f(x) = \pi^2$.

Solution.

Because π^2 is a constant,

$$f'(x) = \frac{d}{dx}\pi^2 = \boxed{0}.$$

6.1. The Power Rule

Formula

If n is any real number, then

$$\frac{d}{dx}(x^n) = nx^{n-1}$$

provided the expression is defined.

Example 6.2.

Differentiate the function.

$$g(t) = t^{-3/4}$$

Solution.

$$g'(t) = -\frac{3}{4}t^{-3/4-1} = \boxed{-\frac{3}{4}t^{-7/4}}.$$

Example 6.3.

Differentiate the function.

$$F(r) = \frac{1}{r^3}$$

Solution.

Because $F(r) = r^{-\frac{1}{3}}$, we have

$$F'(r) = -\frac{1}{3}r^{-\frac{1}{3}-1} = \boxed{-\frac{1}{3}r^{-\frac{4}{3}}}.$$

6.2. The Constant Multiple Rule

Important

If c is a constant and f is differentiable, then

$$\frac{d}{dx}[cf(x)] = c\frac{d}{dx}f(x)$$

Example 6.4.

Differentiate the function.

$$h(x) = 7x^5$$

Solution.

$$h'(x) = 7\frac{d}{dx}x^5 = 7(5x^4) = \boxed{35x^4}.$$

6.3. The Sum and Difference Rules

The Sum and Difference Rules

If f and g are differentiable, then

$$\frac{d}{dx}[f(x) + g(x)] = \frac{d}{dx}f(x) + \frac{d}{dx}g(x)$$

and

$$\frac{d}{dx}[f(x) - g(x)] = \frac{d}{dx}f(x) - \frac{d}{dx}g(x)$$

Example 6.5.

Differentiate the function.

$$p(x) = 4x^3 - 2x^2 + 7x - 5$$

Solution.

$$\begin{aligned} p'(x) &= \frac{d}{dx}(4x^3) + \frac{d}{dx}(-2x^2) + \frac{d}{dx}(7x) + \frac{d}{dx}(-5) \\ &= 12x^2 - 4x + 7 - 0 \\ &= 12x^2 - 4x + 7. \end{aligned}$$

$$\boxed{p'(x) = 12x^2 - 4x + 7}$$

6.4. The Product Rule

The Product Rule

If f and g are differentiable, then

$$\frac{d}{dx}[f(x)g(x)] = f(x)\frac{d}{dx}[g(x)] + g(x)\frac{d}{dx}[f(x)]$$

In prime notation,

$$(fg)' = fg' + gf'$$

Example 6.6.

Differentiate the function.

$$h(x) = 3x(2x+1).$$

Solution.

$$\begin{aligned} h'(x) &= \left[\frac{d}{dx}(3x)\right] \cdot (2x+1) + 3x\left[\frac{d}{dx}(2x+1)\right] \\ &= 3(2x+1) + 3x(2+0) \\ &= 6x + 3 + 6x \\ &= 12x + 3. \end{aligned}$$

$$\boxed{h'(x) = 12x + 3}$$

Example 6.7.

Differentiate the function.

$$g(x) = x^2(1-2x)$$

Solution.

$$\begin{aligned} g'(x) &= \left[\frac{d}{dx}(x^2)\right] \cdot (1-2x) + x^2\left[\frac{d}{dx}(1-2x)\right] \\ &= 2x(1-2x) + x^2(0-2) \\ &= 2x - 4x^2 - 2x^2 \\ &= 2x - 6x^2. \end{aligned}$$

$$\boxed{g'(x) = 2x - 6x^2}$$

Example 6.8.

Differentiate the function.

$$H(u) = (3u - 1)(u + 2)$$

Solution.

$$\begin{aligned} H'(u) &= \left[\frac{d}{du}(3u - 1)\right](u + 2) + (3u - 1)\frac{d}{du}(u + 2) \\ &= (3 - 0)(u + 2) + (3u - 1)(1 + 0) \\ &= 3u + 6 + 3u - 1 \\ &= 6u + 5. \end{aligned}$$

$$\boxed{H'(u) = 6u + 5}$$

Example 6.9.

Find the derivative in two ways.

$$f(x) = (1 + 2x^2)(x - x^2)$$

(a) Use the Product Rule.

(b) Multiply first, then differentiate.

Do your answers agree?

Solution.

(a) By the product rule we get

$$\begin{aligned} f'(a) &= \left[\frac{d}{dx}(1 + 2x^2)\right](x - x^2) + (1 + 2x^2)\frac{d}{dx}(x - x^2) \\ &= (0 + 4x)(x - x^2) + (1 + 2x^2)(1 - 2x) \\ &= 4x^2 - 4x^3 + 1 - 2x + 2x^2 - 4x^3 \\ &= 1 - 2x + 6x^2 - 8x^3. \end{aligned}$$

$$\boxed{f'(a) = 1 - 2x + 6x^2 - 8x^3}$$

(b) By multiplying first, we get

$$f(x) = x - x^2 + 2x^3 - 2x^4.$$

Thus,

$$\begin{aligned} f'(x) &= \frac{d}{dx}(x - x^2 + 2x^3 - 2x^4) \\ &= 1 - 2x + 6x^2 = 8x^3. \end{aligned}$$

$$\boxed{f'(a) = 1 - 2x + 6x^2 - 8x^3}$$

$$\boxed{\text{Yes, the two answers agree.}}$$

6.5. The Quotient Rule

The Quotient Rule

If f and g are differentiable, then

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)\frac{d}{dx}[f(x)] - f(x)\frac{d}{dx}[g(x)]}{[g(x)]^2}$$

In prime notation,

$$\left(\frac{f}{g}\right)' = \frac{gf' - fg'}{g^2}$$

Example 6.10.

Differentiate the function.

$$y = \frac{x^2+1}{x^3-1}$$

Solution.

$$\begin{aligned} \frac{dy}{dx} &= \frac{\left[\frac{d}{dx}(x^2+1)\right](x^3-1) - (x^2+1)\frac{d}{dx}(x^3-1)}{(x^3-1)^2} \\ &= \frac{(2x+0)(x^3-1) - (x^2+1)(3x^2-0)}{(x^3-1)^2} \\ &= \frac{2x(x^3-1) - 3x^2(x^2+1)}{(x^3-1)^2} \\ &= \frac{-x^4 - 2x - 3x^2}{(x^3-1)^2}. \end{aligned}$$

Thus,

$$\boxed{\frac{dy}{dx} = -\frac{2x + 3x^2 + x^4}{(x^3 - 1)^2}}$$

Example 6.11.

Differentiate the function.

$$f(x) = \frac{x}{x + \frac{7}{x}}$$

Solution. First, we re-write $f(x)$ as

$$f(x) = \frac{x^2}{x^2 + 7}.$$

Then,

$$\begin{aligned}\frac{dy}{dx} &= \frac{\left[\frac{d}{dx}(x^2)\right](x^2 + 7) - x^2 \frac{d}{dx}(x^2 + 7)}{(x^2 + 7)^2} \\ &= \frac{2x(x^2 + 7) - x^2(2x + 0)}{(x^2 + 7)^2}\end{aligned}$$

Thus,

$$\boxed{\frac{dy}{dx} = \frac{14x}{(x^2 + 7)^2}}$$

6.6. Applications

Example 6.12.

Find an equation of the tangent line to the curve at the given point.

$$y = \frac{2x}{x + 1}, \qquad (1, 1)$$

Solution.

By quotient rule,

$$\begin{aligned}\frac{dy}{dx} &= \frac{\left[\frac{d}{dx}(2x)\right](x + 1) - 2x \frac{d}{dx}(x + 1)}{(x + 1)^2} \\ &= \frac{2(x + 1) - 2x(1 + 0)}{(x + 1)^2}\end{aligned}$$

$$= \frac{2}{(x+1)^2}.$$

The slope of the tangent line at $(0,0)$ is

$$\left.\frac{dy}{dx}\right|_{x=1} = \frac{2}{(1+1)^2} = \frac{2}{4} = \frac{1}{2}.$$

Thus, the equation of the tangent line is: $\boxed{y - 1 = \frac{1}{2}(x-1)}$.
Equivalently,

$$\boxed{y = \frac{1}{2}x + \frac{1}{2}}$$

Example 6.13.
The equation of motion of a particle is

$$s(t) = t^3 - 3t$$

where s is in meters and t is in seconds.
Find:

(a) the velocity and acceleration functions,

(b) the acceleration after 2 seconds,

(c) the acceleration when the velocity is 0.

Solution.

(a)
$$\boxed{v(t) = 3t^2 - 3} \qquad \text{and} \qquad \boxed{a(t) = 6t}$$

(b)
$$a(2) = 6 \cdot 2 = \boxed{12}.$$

(c). By solving $v(t) = 0$ for $t > 0$ we get $t = 1$. Then,

$$a(1) = 6(1) = \boxed{6}.$$

Additional Examples

Example 6.14.

Determine the numbers at which the function is differentiable.

$$g(x) = |x^2 - 4|$$

Then:

(a) Find a formula for $g'(x)$.

(b) Sketch the graphs of g and g'.

Solution.

Notice that

$$g(x) = \begin{cases} x^2 - 4, & x^2 - 4 \geq 0 \\ -x^2 + 4, & x^2 - 4 < 0. \end{cases}$$

Equivalently,

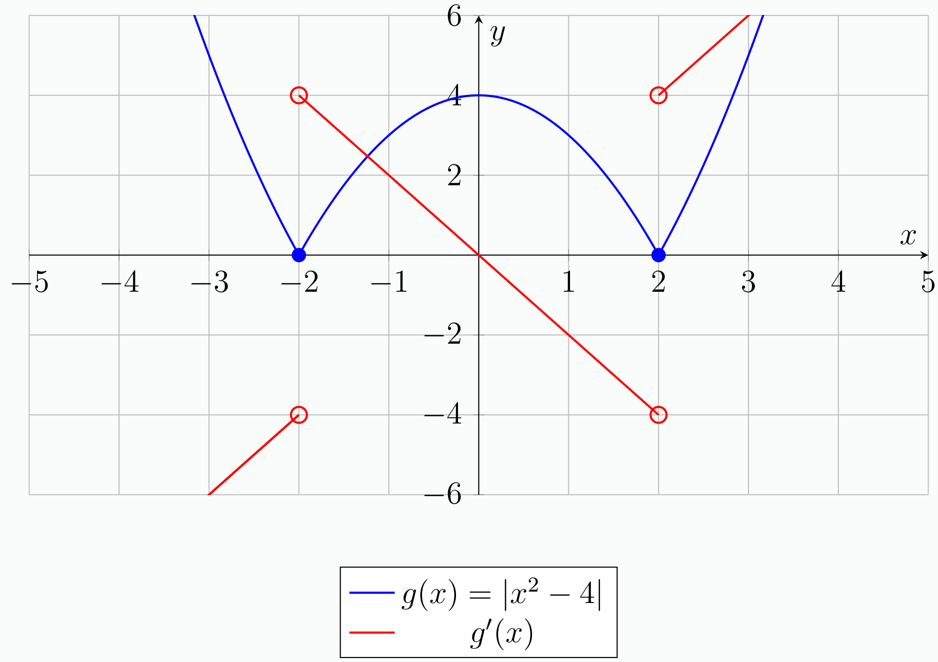
$$g(x) = \begin{cases} x^2 - 4, & x \in (-\infty, -2] \cup [2, \infty) \\ -x^2 + 4, & x \in (-2, 2). \end{cases}$$

So, $g(x)$ is differentiable everywhere except at $x = \pm 2$.

(a)

$$g'(x) = \begin{cases} 2x, & x \in (-\infty, -2) \cup (2, \infty) \\ -2x, & x \in (-2, 2). \end{cases}$$

(b)



Example 6.15.

Find the values of a and b such that the line

$$3x - y = b$$

is tangent to the parabola

$$y = ax^2$$

when $x = 1$.

Solution.

By rewriting the equation of the line as $y = 3x - b$, we see that the slope of the line is 3. The slope of the tangent line to the parabola at $x = 1$ is

$$\left.\frac{dy}{dx}\right|_{x=1} = 2ax|_{x=1} = 2a(1) = 2a.$$

Thus, if the given line is tangent to the curve, we must have

$$2a = 3, \qquad \text{which implies} \qquad a = \frac{3}{2}.$$

Furthermore, because the given line is tangent to the parabola at $x = 1$, it must touches the parabola at $x = 1$.

Thus, we must have

$$y = 3(1) - b = 3 - b \qquad \text{and} \qquad y = a(1)^2 = \frac{3}{2}.$$

By solving this simultaneously we get

$$b = 3 - \frac{3}{2} = \frac{3}{2}.$$

Hence,

$$\boxed{a = b = \frac{3}{2}}$$

Exercises

1. Differentiate:

(i).
$$f(x) = 9x^4 - 5x^2 + 7$$

(ii).
$$g(t) = 4t^{-3} + 6t^{1/2}$$

(iii).
$$h(x) = 5x^3 - 2x^2 + 8x - 1$$

(iv).
$$f(x) = x^2(3 - 4x)$$

(v).
$$g(t) = (2t + 1)(t^2 - 5)$$

(vi).
$$y = \frac{x^2 + 3}{x^2 - 1}$$

(vii).
$$f(x) = \frac{x}{x + \frac{2}{x}}$$

(viii).
$$y = \frac{x^2 + 2x + 1}{\sqrt{x}}$$

(ix).
$$u = \left(\frac{1}{t} + \frac{1}{\sqrt{t}}\right)^2$$

2. Find the derivative in two ways:

$$f(x) = (1 + x^2)(x - 3x^2)$$

3. Find an equation of the tangent line to the curve

$$y = \frac{3x}{x + 2}$$

at the point

$$\left(2, \frac{3}{2}\right).$$

4. Determine the numbers at which the function is differentiable.

$$g(x) = |x^2 - 9|$$

5. Find all tangent lines to the curve

$$y = \frac{x+3}{x-2}$$

that are parallel to the line

$$x - y = 5.$$

6. Determine the numbers at which the function is differentiable.

$$f(x) = \begin{cases} x+1, & x \leq -1, \\ x^2, & -1 < x < 3, \\ 5-x, & x \geq 3 \end{cases}$$

Then:

(a) Find a formula for $f'(x)$.

(b) Sketch the graphs of f and f'.

7. A company sells a product at a price of p dollars per item. The number of items sold is given by

$$q = f(p)$$

and the total revenue is

$$R(p) = pf(p).$$

Suppose

$$f(25) = 8000 \qquad \text{and} \qquad f'(25) = -200.$$

(a) Explain the meaning of $f(25) = 8000$.

(b) Explain the meaning of $f'(25) = -200$.

(c) Find $R'(25)$ and interpret its meaning.

Chapter 7

Derivatives of Trigonometric Functions

Derivatives of Trigonometric Functions

7.1. A Fundamental Trigonometric Limit

Formula

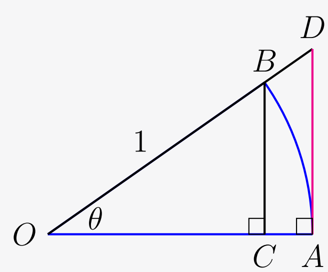
One of the most important limits in calculus is

$$\lim_{\theta \to 0} \frac{\sin\theta}{\theta} = 1.$$

This limit is used in proving the derivatives of the trigonometric functions.

Important

Geometric Illustration



Proof:

It is clear that

$$\text{Area of } OAB\triangle < \text{Area of } \widehat{OAB} < \text{Area of } OAD\triangle.$$

Thus,

$$\frac{1}{2}\sin\theta\cos\theta < \frac{1}{2}\theta < \frac{1}{2}\cdot 1\cdot\tan\theta.$$

By simplification, we get

$$\cos\theta < \frac{\sin\theta}{\theta} < \frac{\theta}{\cos\theta}.$$

Because

$$\lim_{\theta\to 0}\cos\theta = \lim_{\theta\to 0}\frac{\theta}{\cos\theta} = 1,$$

by the squeeze theorem, we get

$$\lim_{\theta\to 0}\frac{\sin\theta}{\theta} = 1.$$

7.2. Limits Involving Trigonometric Functions

Example 7.1.

Evaluate the limit

$$\lim_{\theta\to 0}\frac{\sin 2\theta}{\theta}.$$

Solution.

$$\lim_{\theta\to 0}\frac{\sin 2\theta}{\theta} = \lim_{\theta\to 0}\frac{2\sin 2\theta}{2\theta} = 2\lim_{\theta\to 0}\frac{\sin 2\theta}{2\theta} = 2\cdot 1 = 2.$$

$$\boxed{\lim_{\theta\to 0}\frac{\sin 2\theta}{\theta} = 2}.$$

Example 7.2.

Evaluate the limit

$$\lim_{x\to 0}\frac{\sin 5x}{3x}.$$

Solution.

$$\lim_{x\to 0}\frac{\sin 5x}{3x} = \frac{5}{3}\lim_{x\to 0}\frac{\sin 5x}{5x} = \frac{5}{3}\cdot 1 = \frac{5}{3}.$$

$$\boxed{\lim_{x\to 0}\frac{\sin 5x}{3x} = \frac{5}{3}}.$$

Example 7.3.
Evaluate the limit
$$\lim_{\theta\to 0}\frac{\tan\theta}{\theta}.$$

Solution.

$$\lim_{\theta\to 0}\frac{\tan\theta}{\theta} = \lim_{\theta\to 0}\frac{\sin\theta}{\theta\cos\theta} = \lim_{\theta\to 0}\frac{\sin\theta}{\theta}\cdot\lim_{\theta\to 0}\frac{1}{\cos\theta} = 1\left(\frac{1}{\cos 0}\right) = 1.$$

$$\boxed{\lim_{\theta\to 0}\frac{\tan\theta}{\theta} = 1}.$$

Example 7.4.
Evaluate the limit
$$\lim_{t\to 0}\frac{\tan 6t}{\sin 2t}.$$

Solution.

$$\lim_{t\to 0}\frac{\tan 6t}{\sin 2t} = \frac{6}{2}\lim_{t\to 0}\left(\frac{\tan 6t}{6t}\cdot\frac{2t}{\sin 2t}\right) = 3\left(\lim_{t\to 0}\frac{\tan 6t}{6t}\right)\left(\lim_{t\to 0}\frac{2t}{\sin 2t}\right) = 3(1)(1) = 3.$$

$$\boxed{\lim_{t\to 0}\frac{\tan 6t}{\sin 2t} = 3.}$$

Example 7.5.
Evaluate the limit
$$\lim_{\theta\to 0}\frac{\theta\sin\theta}{\cos\theta - 1}.$$

Solution.
Recall that

$$\sin\theta \equiv 2\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right) \qquad \text{and} \qquad \cos\theta \equiv 1 - 2\sin^2\left(\frac{\theta}{2}\right).$$

Thus,

$$\begin{aligned}\lim_{\theta\to 0}\frac{\theta\sin\theta}{\cos\theta - 1} &= \lim_{\theta\to 0}\frac{\theta\cdot 2\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right)}{-2\sin^2\left(\frac{\theta}{2}\right)}\\ &= -\lim_{\theta\to 0}\frac{\theta\cdot\cos\left(\frac{\theta}{2}\right)}{\sin\left(\frac{\theta}{2}\right)}\end{aligned}$$

$$
\begin{aligned}
&= -\lim_{\theta\to 0}\left(\frac{\cos\left(\frac{\theta}{2}\right)}{1}\cdot\frac{\theta}{\sin\left(\frac{\theta}{2}\right)}\right)\\
&= -2\lim_{\theta\to 0}\left(\frac{\cos\left(\frac{\theta}{2}\right)}{1}\cdot\frac{\frac{\theta}{2}}{\sin\left(\frac{\theta}{2}\right)}\right)
\end{aligned}
$$

$$
\begin{aligned}
&= -2\lim_{\theta\to 0}\left(\frac{\cos\left(\frac{\theta}{2}\right)}{1}\right)\lim_{\theta\to 0}\left(\frac{\frac{\theta}{2}}{\sin\left(\frac{\theta}{2}\right)}\right)\\
&= -2(1)(1)\\
&= -2.
\end{aligned}
$$

$$\boxed{\lim_{\theta\to 0}\frac{\theta\sin\theta}{\cos\theta-1}=-2.}$$

Example 7.6.
Evaluate the limit
$$\lim_{x\to 0}\frac{\sin(3x)}{5x^3-4x}.$$

Solution.

$$
\begin{aligned}
\lim_{x\to 0}\frac{\sin(3x)}{5x^3-4x} &= \lim_{x\to 0}\frac{\sin(3x)}{x(5x^2-4)}\\
&= \left(\lim_{x\to 0}\frac{3\sin(3x)}{3x}\right)\left(\lim_{x\to 0}\frac{1}{5x^2-4}\right)\\
&= (3)\left(\frac{1}{0-4}\right)\\
&= -\frac{3}{4}.
\end{aligned}
$$

$$\boxed{\lim_{x\to 0}\frac{\sin(3x)}{5x^3-4x}=-\frac{3}{4}.}$$

Example 7.7.
Evaluate the limit
$$\lim_{x\to 0}\frac{\sin(x^2)}{x}.$$

Solution.

$$\begin{aligned}\lim_{x\to 0}\frac{\sin(x^2)}{x} &= \lim_{x\to 0}\frac{x\sin(x^2)}{x^2}\\ &= \left(\lim_{x\to 0} x\right)\left(\lim_{x\to 0}\frac{\sin(x^2)}{x^2}\right)\\ &= (0)(1)\end{aligned}$$

$$\boxed{\lim_{x\to 0}\frac{\sin(x^2)}{x} = 0.}$$

7.3. Derivatives of Trigonometric Functions

Formula

$\frac{d}{dx}(\sin x) = \cos x$	$\frac{d}{dx}(\csc x) = -\csc x\cot x$
$\frac{d}{dx}(\cos x) = -\sin x$	$\frac{d}{dx}(\sec x) = \sec x\tan x$
$\frac{d}{dx}(\tan x) = \sec^2 x$	$\frac{d}{dx}(\cot x) = -\csc^2 x$

Example 7.8.
Differentiate the function

$$f(x) = x^2\sin x.$$

Solution. We use the product rule directly:

$$\begin{aligned}f'(x) &= \left[\frac{d}{dx}(x^2)\right]\sin x + x^2\frac{d}{dx}(\sin x)\\ &= 2x\sin x + x^2\cos x.\end{aligned}$$

$$\boxed{f'(x) = 2x\sin x + x^2\cos x.}$$

Example 7.9.
Differentiate the function

$$f(x) = x\cos x + 3\tan x.$$

Solution.

$$\begin{aligned} f'(x) &= 1 \cdot \cos x + x(-\sin x) + 3\sec^2 x \\ &= \cos x - x\sin x + 3\sec^2 x. \end{aligned}$$

$$\boxed{f'(x) = \cos x - x\sin x + 3\sec^2 x.}$$

Example 7.10.
Differentiate the function

$$y = u(a\cos u + b\cot u).$$

Solution.

$$\begin{aligned} \frac{dy}{du} &= \left[\frac{du}{du}\right](a\cos u + b\cot u) + u\frac{d}{du}(a\cos u + b\cot u) \\ &= 1 \cdot (a\cos u + b\cot u) + u(-a\sin u - b\csc^2 u) \\ &= a(\cos u - u\sin u) + b(\cot u - csc^2 u). \end{aligned}$$

$$\boxed{\frac{dy}{du} = a(\cos u - u\sin u) + b(\cot u - csc^2 u).}$$

Example 7.11.
Differentiate the function

$$y = \frac{x}{2 - \tan x}.$$

Solution.
We use the quotient rule.

$$\begin{aligned} \frac{dy}{dx} &= \frac{\left[\frac{d}{dx}(x)\right](2 - \tan x) - x\frac{d}{dx}(2 - \tan x)}{(2 - \tan x)^2} \\ &= \frac{1 \cdot (2 - \tan x) - x(0 - \sec^2 x)}{(2 - \tan x)^2} \\ &= \frac{2 - \tan x + x\sec^2 x}{(2 - \tan x)^2}. \end{aligned}$$

$$\boxed{\frac{dy}{dx} = \frac{2 - \tan x + x\sec^2 x}{(2 - \tan x)^2}}$$

Example 7.12.

Find an equation of the tangent line to the curve

$$y = \sin x + \cos x$$

at the point $(0, 1)$.

Solution. The slope of the tangent line is

$$\begin{aligned} \left.\frac{dy}{dx}\right|_{x=0} &= \left.(\cos x - \sin x)\right|_{x=0} \\ &= \cos 0 - \sin 0 \\ &= 1 - 0 \\ &= 1. \end{aligned}$$

Therefore, the equation of the tangent line is

$$\boxed{y - 1 = 1(x - 0)}$$

Equivalently,

$$\boxed{y = x + 1}$$

Example 7.13.

If

$$H(\theta) = \theta \sin \theta,$$

find $H'(\theta)$ and $H''(\theta)$.

Solution.

$$\begin{aligned} H'(\theta) &= \left[\frac{d}{d\theta}(\theta)\right] \sin \theta + \theta \frac{d}{d\theta}(\sin \theta) \\ &= 1 \cdot \sin \theta + \theta \cos \theta \\ &= \sin \theta + \theta \cos \theta. \end{aligned}$$

$$\begin{aligned} H''(\theta) &= \frac{d}{d\theta} H'(\theta) \\ &= \frac{d}{d\theta}(\sin \theta + \theta \cos \theta) \\ &= \cos \theta + \cos \theta + \theta(-\sin \theta) \\ &= 2\cos \theta - \theta \sin \theta. \end{aligned}$$

Thus,

$$\boxed{\begin{aligned} H'(\theta) &= \sin \theta + \theta \cos \theta \\ H''(\theta) &= 2\cos \theta - \theta \sin \theta \end{aligned}}$$

Example 7.14.

Suppose

$$f\left(\frac{\pi}{4}\right) = 2, \qquad f'\left(\frac{\pi}{4}\right) = 5,$$

and let

$$g(x) = f(x)\sin x, \qquad h(x) = \frac{\cos x}{f(x)}.$$

Find:

(a) $g'\left(\frac{\pi}{4}\right)$

(b) $h'\left(\frac{\pi}{4}\right)$

Solution.

(a) By the product rule, we get

$$g'(x) = f'(x)\sin x + f(x)\cos x.$$

Thus,

$$\begin{aligned} g'\left(\frac{\pi}{4}\right) &= f'\left(\frac{\pi}{4}\right)\sin\left(\frac{\pi}{4}\right) + f\left(\frac{\pi}{4}\right)\cos\left(\frac{\pi}{4}\right) \\ &= 5 \cdot \frac{1}{\sqrt{2}} + 2 \cdot \frac{1}{\sqrt{2}} \\ &= \frac{7}{\sqrt{2}}. \end{aligned}$$

(b) By the quotient rule, we get

$$h'(x) = \frac{-f(x)\sin x - f'(x)\cos x}{[f(x)]^2}.$$

Thus,

$$\begin{aligned} h'\left(\frac{\pi}{4}\right) &= -\frac{f\left(\frac{\pi}{4}\right)\sin\left(\frac{\pi}{4}\right) + f'\left(\frac{\pi}{4}\right)\cos\left(\frac{\pi}{4}\right)}{[f\left(\frac{\pi}{4}\right)]^2} \\ &= -\frac{2 \cdot \frac{1}{\sqrt{2}} + 5 \cdot \frac{1}{\sqrt{2}}}{2^2} \\ &= -\frac{7}{4\sqrt{2}}. \end{aligned}$$

Thus,

$$\boxed{\begin{aligned} g'\left(\frac{\pi}{4}\right) &= \frac{7}{\sqrt{2}} \\ h'\left(\tfrac{\pi}{4}\right) &= -\tfrac{7}{4\sqrt{2}} \end{aligned}}$$

Example 7.15.

For what values of x does the graph of

$$f(x) = x + 2\sin x$$

have a horizontal tangent?

Solution.

We have

$$f'(x) = 1 + 2\cos x.$$

Horizontal tangents have the slope equals to 0. Thus, by solving the equation $f'(x) = 0$ for x we can find the x values at where the graph of $f(x)$ has horizontal tangents.

$$\begin{aligned} f'(x) &= 0 \\ 1 + 2\cos x &= 0 \\ \cos x &= -\frac{1}{2} \\ \cos x &= \cos\left(\frac{2\pi}{3}\right) \\ x &= 2n\pi \pm \frac{2\pi}{3}, \quad n \in \mathbb{Z}. \end{aligned}$$

Thus,

$$\boxed{x = 2n\pi \pm \frac{2\pi}{3}, \quad n \in \mathbb{Z}}$$

Example 7.16.

Determine all values of x for which the graph of

$$g(x) = x^2 - 3\cos x$$

has a horizontal tangent.

Solution.
We have

$$g'(x) = 2x + 3\sin x.$$

Horizontal tangents have the slope equals to 0. Thus, by solving the equation $g'(x) = 0$ for x we can find the x values at where the graph of $g(x)$ has horizontal tangents.

$$\begin{aligned} g'(x) &= 0 \\ 2x + 3\sin x &= 0 \\ \sin x &= -\frac{2}{3}x. \end{aligned}$$

Because $-1 \leq \sin x \leq 1$, we get

$$-1 \leq -\frac{2}{3}x \leq 1.$$

This implies

$$-\frac{2}{3} \leq x \leq \frac{2}{3}.$$

This says that the equation $g'(x) = 0$ has solutions only on the interval $\left[-\frac{2}{3}, \frac{2}{3}\right]$.
Notice that on $\left[-\frac{2}{3}, 0\right)$ we have $-3\sin x > 0$ and $2x < 0$.
Similarly on $\left(0, \frac{2}{3}\right]$ we have $-3\sin x < 0$ and $2x > 0$.
So, there are no solutions to the equation $g'(x) = 0$, or equivalently $2x = -\sin x$, on $\left[-\frac{2}{3}, 0\right) \cup \left(0, \frac{2}{3}\right]$.
However, we have

$$g'(0) = 2(0) + 3\sin 0 = 0.$$

Thus, $x = 0$ is the only solution to the equation $g'(x) = 0$.

Hence, the graph of $g(x)$ has horizontal tangents only at

$$\boxed{x = 0}.$$

Exercises

1. Evaluate.

(i).
$$\lim_{x\to 0}\frac{\sin(7x)}{x}$$

(ii).
$$\lim_{x\to 0}\frac{\tan(4x)}{x}$$

(iii).
$$\lim_{x\to 0}\frac{1-\cos x}{x^2}$$

(iv).
$$\lim_{x\to 0}\frac{\sin(2x)}{\tan(5x)}$$

(v).
$$\lim_{x\to 0}\frac{\sin(x^3)}{x}$$

(vi).
$$\lim_{x\to \pi/3}\frac{\sin x-\sqrt{3}/2}{x-\pi/3}$$

2. Differentiate:

(i).
$$f(x)=x^3\cos x$$

(ii).
$$y=x\tan x-\sec x$$

(iii).
$$y=\frac{\sin x}{1+\cos x}$$

(iv).
$$y=\frac{x\cos x}{1+x}$$

3. Find an equation of the tangent line to

$$y=\sin x-x\cos x$$

at the point $(0,0)$.

4. If
$$G(t) = t\cos t,$$
find $G'(t)$ and $G''(t)$.

5. Suppose
$$f\left(\frac{\pi}{6}\right) = 3, \qquad f'\left(\frac{\pi}{6}\right) = 1.$$
If
$$g(x) = f(x)\cos x,$$
find
$$g'\left(\frac{\pi}{6}\right).$$

6. Find all values of x where the graph of
$$f(x) = 2x - \sin x$$
has a horizontal tangent.

Chapter 8

The Chain Rule

8.1. The Chain Rule

The Chain Rule

If g is differentiable at x and f is differentiable at $g(x)$, then the composite function

$$F(x) = f(g(x))$$

is differentiable and

$$F'(x) = f'(g(x)) \cdot g'(x).$$

In Leibniz notation, if

$$y = f(u) \qquad \text{and} \qquad u = g(x),$$

then

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}.$$

Important

Power Rule Combined with the Chain Rule

If $u = g(x)$ is differentiable and n is any real number, then

$$\frac{d}{dx}(u^n) = nu^{n-1}\frac{du}{dx}.$$

Equivalently,

$$\frac{d}{dx}\left([g(x)]^n\right) = n[g(x)]^{n-1}g'(x).$$

Example 8.1.
Differentiate

$$y = \sqrt[3]{1+4x}.$$

Solution.
By re-writing in the exponential notation we get $y = (1+4x)^{\frac{1}{3}}$.
Thus,

$$\begin{aligned}\frac{dy}{dx} &= \frac{1}{3}(1+4x)^{\frac{1}{3}-1} \cdot \frac{d}{dx}(1+4x)\\ &= \frac{1}{3}(1+4x)^{-\frac{2}{3}}(0+4)\\ &= \frac{4}{3}(1+4x)^{-\frac{2}{3}}.\end{aligned}$$

$$\boxed{\frac{dy}{dx} = \frac{4}{3}(1+4x)^{-\frac{2}{3}}}$$

Example 8.2.
Differentiate

$$y = (2x^3+5)^4.$$

Solution.

$$\begin{aligned}\frac{dy}{dx} &= 4(2x^3+5)^{4-1}\frac{d}{dx}(2x^3+5)\\ &= 4(2x^3+5)^3(6x^2+0)\\ &= 24x^2(2x^3+5)^3.\end{aligned}$$

$$\boxed{\frac{dy}{dx} = 24x^2(2x^3+5)^3}$$

Example 8.3.
Differentiate

$$y = \tan(\pi x).$$

Solution.

$$\begin{aligned}\frac{dy}{dx} &= \sec^2(\pi x)\frac{d}{dx}(\pi x)\\ &= \sec^2(\pi x)\cdot\pi\end{aligned}$$

$$\boxed{\frac{dy}{dx} = \pi\sec^2(\pi x)}$$

Example 8.4.

Differentiate

$$y = \sin(\cot x).$$

Solution.

$$\begin{aligned}\frac{dy}{dx} &= \cos(\cot x)\frac{d}{dx}(\cot x)\\ &= \cos(\cot x)\cdot(-\csc^x)\end{aligned}$$

$$\boxed{\frac{dy}{dx} = -\csc^2 x\cos(\cot x)}$$

Example 8.5.

Differentiate

$$y = \sqrt{\sin x}.$$

Solution. By re-writing in the exponential notation we get $y = (\sin x)^{\frac{1}{2}}$. Thus,

$$\begin{aligned}\frac{dy}{dx} &= \frac{1}{2}(\sin x)^{\frac{1}{2}-1}\frac{d}{dx}(\sin x)\\ &= \frac{1}{2}(\sin x)^{-\frac{1}{2}}\cos x.\end{aligned}$$

$$\boxed{\frac{dy}{dx} = \frac{1}{2}(\sin x)^{-\frac{1}{2}}\cos x}$$

Example 8.6.

Differentiate

$$g(x) = (2-\sin x)^{3/2}.$$

Solution.

$$\begin{aligned} g'(x) &= \frac{3}{2}(2-\sin x)^{3/2-1}\frac{d}{dx}(2-\sin x) \\ &= \frac{3}{2}(2-\sin x)^{\frac{1}{2}}(0-\cos x). \end{aligned}$$

$$\boxed{g'(x) = -\frac{3}{2}\cos x(2-\sin x)^{\frac{1}{2}}}$$

Example 8.7.
Differentiate

$$A(t) = \frac{1}{(\cos t + \tan t)^2}.$$

Solution.
We have

$$A(t) = (\cos t + \tan t)^{-2}.$$

Thus,

$$\begin{aligned} A'(t) &= -2(\cos t + \tan t)^{-2-1}\frac{d}{dt}(\cos t + \tan t) \\ &= -2(\cos t + \tan t)^{-3}(-\sin t + \sec^2 t) \end{aligned}$$

$$\boxed{A'(t) = -2(\cos t + \tan t)^{-3}(-\sin t + \sec^2 t)}$$

Example 8.8.
Differentiate

$$f(x) = \frac{1}{\sqrt[3]{x^2-1}}.$$

Solution.
We have

$$f(x) = (x^2-1)^{-\frac{1}{3}}.$$

Thus,

$$\begin{aligned} f'(x) &= -\frac{1}{3}(x^2-1)^{-\frac{1}{3}-1}\frac{d}{dx}(x^2-1) \\ &= -\frac{1}{3}(x^2-1)^{-\frac{4}{3}}(2x-0) \end{aligned}$$

$$\boxed{f'(x) = -\frac{2}{3}x(x^2-1)^{-\frac{4}{3}}}$$

Example 8.9.

Differentiate

$$f(\theta) = \cos(\theta^2).$$

Solution.

$$\begin{aligned} f'(\theta) &= -\sin(\theta^2)\frac{d}{d\theta}(\theta^2) \\ &= -\sin(\theta^2)\cdot 2\theta \end{aligned}$$

$$\boxed{f'(\theta) = -2\theta\sin(\theta^2)}$$

Example 8.10.

Differentiate

$$f(x) = (2x-3)^4(x^2+x+1)^5.$$

Solution.

$$\begin{aligned} f'(x) &= \left[\frac{d}{dx}(2x-3)^4\right](x^2+x+1)^5 + (2x-3)^4\frac{d}{dx}(x^2+x+1)^5 \\ &= [4(2x-3)^{4-1}(2-0)](x^2+x+1)^5 + (2x-3)^4[5(x^2+x+1)^{5-1}(2x+1+0)] \\ &= 8(2x-3)^3(x^2+x+1)^5 + 5(2x+1)(2x-3)^4(x^2+x+1)^4. \end{aligned}$$

$$\boxed{f'(x) = 8(2x-3)^3(x^2+x+1)^5 + 5(2x+1)(2x-3)^4(x^2+x+1)^4}$$

Example 8.11.

Differentiate

$$g(x) = (x^2+1)^3(x^2+2)^6.$$

Solution.

$$\begin{aligned} g'(x) &= \left[\frac{d}{dx}(x^2+1)^3\right](x^2+2)^6 + (x^2+1)^3\frac{d}{dx}(x^2+2)^6 \\ &= [3(x^2+1)^{3-1}(2x+0)](x^2+2)^6 + (x^2+1)^3[6(x^2+2)^{6-1}(2x+0)] \\ &= 6x(x^2+1)^2(x^2+2)^6 + 12x(x^2+1)^3(x^2+2)^5. \end{aligned}$$

$$\boxed{g'(x) = 6x(x^2+1)^2(x^2+2)^6 + 12x(x^2+1)^3(x^2+2)^5}$$

Example 8.12.
Differentiate

$$y = \left(x+\frac{1}{x}\right)^5.$$

Solution.

$$\begin{aligned} \frac{dy}{dx} &= 5\left(x+\frac{1}{x}\right)^{5-1}\frac{d}{dx}\left(x+\frac{1}{x}\right) \\ &= 5\left(x+\frac{1}{x}\right)^4\left(1-\frac{1}{x^2}\right). \end{aligned}$$

$$\boxed{\frac{dy}{dx} = 5\left(x+\frac{1}{x}\right)^4\left(1-\frac{1}{x^2}\right)}$$

Example 8.13.
Differentiate

$$y = \sqrt{\frac{x}{x+1}}.$$

Solution. We have

$$y = \left(\frac{x}{x+1}\right)^{\frac{1}{2}}.$$

Thus,

$$\begin{aligned} \frac{dy}{dx} &= \frac{1}{2}\left(\frac{x}{x+1}\right)^{\frac{1}{2}-1}\frac{d}{dx}\left(\frac{x}{x+1}\right) \\ &= \frac{1}{2}\left(\frac{x}{x+1}\right)^{-\frac{1}{2}}\cdot\frac{1(x+1)-x(1+0)}{(x+1)^2}. \end{aligned}$$

$$\boxed{\frac{dy}{dx} = \frac{1}{2(x+1)^2}\left(\frac{x}{x+1}\right)^{-\frac{1}{2}}}$$

Example 8.14.

Differentiate

$$h(\theta) = \tan(\theta^2 \sin\theta).$$

Solution.

$$\begin{aligned} h'(\theta) &= \sec^2(\theta^2 \sin\theta)\frac{d}{d\theta}(\theta^2 \sin\theta) \\ &= \sec^2(\theta^2 \sin\theta) \cdot [2\theta \sin\theta + \theta^2 \cos\theta] \end{aligned}$$

$$\boxed{h'(\theta) = \theta(2\sin\theta + \theta\cos\theta)\sec^2(\theta^2 \sin\theta)\cdot}$$

Example 8.15.

Differentiate

$$y = \cot^2(\sin\theta).$$

Solution.

$$\begin{aligned} \frac{dy}{d\theta} &= 2\cot(\sin\theta)\frac{d}{d\theta}\sin\theta \\ &= 2\cot(\sin\theta)\cos\theta \end{aligned}$$

$$\boxed{\frac{dy}{d\theta} = 2\cot(\sin\theta)\cos\theta}$$

Example 8.16.

Differentiate

$$f(t) = \tan(\sec(\cos t)).$$

Solution.

$$\begin{aligned} f'(t) &= \sec^2(\sec(\cos t))\frac{d}{dt}(\sec(\cos t)) \\ &= \sec^2(\sec(\cos t))[\sec(\cos t)\tan(\cos t)(-\sin t)] \\ &= \sec^2(\sec(\cos t))[-\sin t\sec(\cos t)\tan(\cos t)] \end{aligned}$$

$$\boxed{f'(t) = -\sin t \sec(\cos t)\tan(\cos t)\sec^2(\sec(\cos t))}$$

Example 8.17.

Differentiate

$$g(u) = \left[(u^2-1)^6 - 3u\right]^4.$$

Solution.

$$\begin{aligned} g'(u) &= 4\left[(u^2-1)^6 - 3u\right]^{4-1} \cdot \frac{d}{du}\left[(u^2-1)^6 - 3u\right] \\ &= 4\left[(u^2-1)^6 - 3u\right]^3 \cdot \left[6(u^2-1)^{6-1}(2u-0) - 3\right] \\ &= 4\left[(u^2-1)^6 - 3u\right]^3 \cdot \left[12u(u^2-1)^5 - 3\right] \end{aligned}$$

$$\boxed{g'(u) = 12\left[4u(u^2-1)^5 - 1\right]\left[(u^2-1)^6 - 3u\right]^3 \cdot}$$

Example 8.18.

Differentiate

$$y = \sqrt{x + \sqrt{x}}.$$

Solution.

By re-writing in the exponential notation we get

$$y = \left(x + x^{\frac{1}{2}}\right)^{\frac{1}{2}}.$$

Thus,

$$\begin{aligned} \frac{dy}{dx} &= \frac{1}{2}\left(x + x^{\frac{1}{2}}\right)^{\frac{1}{2}-1} \frac{d}{dx}(x + x^{\frac{1}{2}}) \\ &= \frac{1}{2}\left(x + x^{\frac{1}{2}}\right)^{-\frac{1}{2}} \left(1 + \frac{1}{2}x^{\frac{1}{2}-1}\right) \\ &= \frac{1}{2}\left(x + x^{\frac{1}{2}}\right)^{-\frac{1}{2}} \left(1 + \frac{1}{2}x^{-\frac{1}{2}}\right) \end{aligned}$$

$$\boxed{\frac{dy}{dx} = \frac{2\sqrt{x}+1}{4\sqrt{x}\sqrt{x+\sqrt{x}}}}$$

Tangent Line Problems

Example 8.19.

Find the equation of the tangent line to

$$y = (3x+2)^{-5}$$

at the point $\left(0, \dfrac{1}{32}\right)$.

Solution.

First, lets find the derivative:

$$\begin{aligned}\frac{dy}{dx} &= -5(3x+2)^{-5-1}\frac{d}{dx}(3x+2)\\ &= -5(3x+2)^{-6}(3+0)\\ &= -15(3x+2)^{-6}.\end{aligned}$$

Thus, the slope the tangent line at the point $\left(0, \dfrac{1}{32}\right)$ is

$$\begin{aligned}\left.\frac{dy}{dx}\right|_{x=0} &= -15(3\cdot 0+2)^{-6}\\ &= -15\cdot 3^{-6}\\ &= -\frac{5}{3^5}.\end{aligned}$$

Therefore, an equation of the tangent line is

$$\boxed{y - \frac{1}{32} = -\frac{5}{3^5}(x-0)}$$

Equivalently,

$$\boxed{y = -\frac{5}{243}x + \frac{1}{32}}$$

8.2. Applications of the Chain Rule

Example 8.20.

At what point on the curve

$$y = \sqrt{1+3x}$$

is the tangent line perpendicular to the line

$$3x + y = 1?$$

Solution.

By differentiating $y = \sqrt{1+3x}$ with respect to x we get

$$\begin{aligned}\frac{dy}{dx} &= \frac{1}{2}(1+3x)^{\frac{1}{2}-1}\frac{d}{dx}(1+3x)\\ &= \frac{1}{2}(1+3x)^{-\frac{1}{2}}(0+3)\\ &= \frac{3}{2\sqrt{1+3x}}\end{aligned}$$

This is the slope of the tangent line to the curve $y = \sqrt{1+3x}$ at a generic point (x, y). The slope of the line $3x+y=1$ is -3. So, in order for a tangent line to be perpendicular to this line we must have

$$\frac{3}{2\sqrt{1+3x}} \cdot (-3) = -1.$$

Thus,

$$\begin{aligned}\frac{3}{2\sqrt{1+3x}} \cdot (-3) &= -1\\ \sqrt{1+3x} &= \frac{9}{2}\\ 1+3x &= \left(\frac{9}{2}\right)^2\\ 3x &= \frac{81}{4} - 1\\ 3x &= \frac{77}{4}\\ x &= \frac{77}{12}\end{aligned}$$

$$\boxed{x = \frac{77}{12}}$$

Example 8.21.

Suppose

$$F(x) = f(g(x)),$$

where

$$f'(1) = 4, \qquad g(2) = 1, \qquad g'(2) = 3.$$

Find $F'(2)$.

Solution.

We have

$$F'(x) = f'(g(x))g'(x).$$

Thus,

$$\begin{aligned} F'(2) &= f'(g(2))g'(2) \\ &= f'(1) \cdot 3 \\ &= 4 \cdot 3 \end{aligned}$$

$$\boxed{F'(2) = 12}$$

Example 8.22.

Let

$$r(x) = f(g(h(x))),$$

where

$$h(1) = 2, \qquad h'(1) = 3, \qquad g(2) = 4, \qquad g'(2) = 5, \qquad f'(4) = 6.$$

Find $r'(1)$.

Solution.

We have

$$r'(x) = f'(g(h(x)))g'(h(x))h'(x).$$

Thus,

$$\begin{aligned} r'(1) &= f'(g(h(1)))g'(h(1))h'(1) \\ &= f'(g(2))g'(2) \cdot 3 \\ &= f'(4) \cdot 5 \cdot 3 \\ &= 6 \cdot 5 \cdot 3 \end{aligned}$$

$$\boxed{r'(2) = 90}$$

Example 8.23.
If

$$f(x) = xg(x^2),$$

find $f'(x)$ in terms of g and g'.

Solution.

$$\begin{aligned} f'(x) &= \left[\frac{d}{dx}(x)\right] g(x^2) + x\frac{d}{dx}g(x^2) \\ &= 1 \cdot g(x^2) + x\left[g'(x^2)\frac{d}{dx}(x^2)\right] \\ &= g(x^2) + xg'(x^2) \cdot 2x \end{aligned}$$

$$\boxed{f'(x) = g(x^2) + 2x^2g'(x^2)}$$

Example 8.24.
If

$$f(x) = x^2g(3x+1),$$

find $f'(x)$ in terms of g and g'.

Solution.
Using the Product Rule,

$$\begin{aligned} f'(x) &= \left[\frac{d}{dx}(x^2)\right] g(3x+1) + x^2\frac{d}{dx}\left(g(3x+1)\right) \\ &= 2x\, g(3x+1) + x^2\left[g'(3x+1)\frac{d}{dx}(3x+1)\right] \\ &= 2x\, g(3x+1) + 3x^2g'(3x+1). \end{aligned}$$

$$\boxed{f'(x) = 2x\, g(3x+1) + 3x^2g'(3x+1)}$$

Exercises

Differentiate the following functions.

1. $y = (5x-1)^6$
2. $y = \sqrt{2x+7}$
3. $y = \sin(4x)$
4. $y = \cos(x^3)$
5. $y = \tan(\sqrt{x})$
6. $y = (x^2+1)^{-3}$
7. $y = (3x^2-5x)^7$
8. $y = \sec(x^2)$
9. $y = \sqrt{1+\sin x}$
10. $y = (1+\cos x)^5$
11. $y = \left(x+\dfrac{1}{x}\right)^4$
12. $y = \sqrt{\dfrac{x+1}{x-1}}$
13. $y = \sin(\cos x)$
14. $y = \cot(\sin x)$
15. $y = \tan(\sec x)$

Find the equation of the tangent line.

16. $y = (2x+1)^3$ at $(0, 1)$
17. $y = \sqrt{x+4}$ at $(5, 3)$
18. $y = \sin(x^2)$ at $(0, 0)$

Applications.

19. If
$$F(x) = f(g(x)),$$
where
$$f'(2) = 5, \qquad g(1) = 2, \qquad g'(1) = 4,$$
find $F'(1)$.

20. If
$$r(x) = f(g(h(x))),$$
where
$$h'(1) = 2, \qquad g'(3) = 4, \qquad f'(5) = 6, \qquad h(1) = 3, \qquad g(3) = 5,$$
find $r'(1)$.

Chapter 9

Implicit Differentiation

9.1. Implicit Differentiation

Introduction

Some equations describe a relationship between two variables without explicitly solving for one variable in terms of the other. Such an equation is said to define the variables **implicitly**.

For instance, consider the equation

$$x^2 + y^2 = 25. \tag{9.1}$$

This equation represents a circle of radius 5 centered at the origin. It is not written in the form

$$y = f(x).$$

In fact, if we solve Equation (9.1) for y, we obtain

$$y^2 = 25 - x^2,$$

and hence

$$y = \pm\sqrt{25 - x^2}.$$

Thus, for many values of x, there are two corresponding values of y. For example, when $x = 3$,

$$y = \pm\sqrt{25 - 9} = \pm 4.$$

Therefore, the points

$$(3, 4) \qquad \text{and} \qquad (3, -4)$$

both lie on the circle. Since the same input $x = 3$ produces two different outputs, the entire circle does not define y as a function of x.

Similarly, by solving Equation (9.1) for x we get

$$x = \pm\sqrt{25 - y^2}.$$

Thus, the entire circle does not define x as a function of y either.

However, portions of the circle do define functions. For example, the upper semicircle is given by

$$y = \sqrt{25 - x^2}, \qquad -5 \leq x \leq 5,$$

and the lower semicircle is given by

$$y = -\sqrt{25 - x^2}, \qquad -5 \leq x \leq 5.$$

Thus, although the full circle does not define y as a function of x globally, it does define y as a function of x locally on suitable portions of the curve.

Important

A relation may fail to define one variable as a function of the other on the entire curve, but it may define a function locally near a particular point.

Even when an equation is not explicitly solved for y, we can often find the slope of the curve by differentiating both sides of the equation with respect to x. When doing so, we treat y as a differentiable function of x.

For example, differentiating

$$x^2 + y^2 = 25$$

with respect to x gives

$$\frac{d}{dx}\left(x^2\right) + \frac{d}{dx}\left(y^2\right) = \frac{d}{dx}(25).$$

Since y depends on x, the Chain Rule must be used when differentiating y^2. Therefore,

$$2x + 2y\frac{dy}{dx} = 0.$$

By solving for $\frac{dy}{dx}$, we obtain

$$2y\frac{dy}{dx} = -2x,$$

and hence

$$\boxed{\frac{dy}{dx} = -\frac{x}{y}}.$$

This formula gives the slope of the tangent line to the circle at any point (x, y) on the circle for which $y \neq 0$.

For example, at the point $(3, 4)$,

$$\frac{dy}{dx} = -\frac{3}{4}.$$

At the point $(3, -4)$,

$$\frac{dy}{dx} = -\frac{3}{-4} = \frac{3}{4}.$$

Thus, the upper and lower portions of the circle have different tangent slopes at points with the same x-coordinate.

Important

The method of differentiating an equation involving both x and y, without first solving explicitly for y, is called **implicit differentiation**.

Warning

When using implicit differentiation, remember that y is treated as a function of x. Therefore, every time a term involving y is differentiated, the Chain Rule introduces a factor of $\dfrac{dy}{dx}$.

Steps for Implicit Differentiation

Suppose we want to find $y' \left(= \dfrac{dy}{dx}\right)$.

1. Differentiate both sides with respect to the x.
2. Whenever differentiating a term involving y, multiply by y'.
3. Collect all terms involving y' on one side.
4. Factor out y'.
5. Solve for y'.

Example 9.1.

Find $\dfrac{dy}{dx}$ by implicit differentiation.

$$x^2 - 4xy + y^2 = 4$$

Solution.

$$\begin{aligned}
\frac{d}{dx}(x^2 - 4xy + y^2) &= \frac{d}{dx}(4) \\
2x - 4y - 4x\frac{dy}{dx} + 2y\frac{dy}{dx} &= 0 \\
(4x - 2y)\frac{dy}{dx} &= 2x - 4y \\
\frac{dy}{dx} &= \frac{2x - 4y}{4x - 2y}
\end{aligned}$$

$$\boxed{\frac{dy}{dx} = \frac{x - 2y}{2x - y}}$$

Example 9.2.

Find $\dfrac{dy}{dx}$ by implicit differentiation.

$$\frac{x^2}{x + y} = y^2 + 1$$

Solution.

$$\begin{aligned}
\frac{d}{dx}\left(\frac{x^2}{x + y}\right) &= \frac{d}{dx}(y^2 + 1) \\
\frac{2x(x + y) - x^2(1 + y')}{(x + y)^2} &= 2yy' + 0 \\
x^2 + 2xy - x^2y' &= 2y(x + y)^2y' \\
y'\left[x^2 + 2y(x + y)^2\right] &= x^2 + 2xy \\
y' &= \frac{x^2 + 2xy}{x^2 + 2y(x + y)^2}.
\end{aligned}$$

$$\boxed{\frac{dy}{dx} = \frac{x^2 + 2xy}{x^2 + 2y(x + y)^2}}$$

9.2. Implicit Differentiation with Trigonometric Functions

When trigonometric functions contain y, we still use the Chain Rule.

For example,

$$\frac{d}{dx}(\sin y) = \cos y \cdot y' \quad \text{and} \quad \frac{d}{dx}(\cos(xy)) = -\sin(xy)\frac{d}{dx}(xy).$$

Example 9.3.
Find $\dfrac{dy}{dx}$ by implicit differentiation.

$$\cos(xy) = 1 + \sin y$$

Solution.

$$\begin{aligned}
\frac{d}{dx}[\cos(xy)] &= \frac{d}{dx}(1 + \sin y) \\
-\sin(xy)[1 \cdot y + xy'] &= 0 + \cos y \cdot y' \\
-y\sin(xy) + xy'\sin(xy) &= y'\cos y \\
(x\sin(xy) - \cos y)y' &= y\sin(xy) \\
y' &= \frac{y\sin(xy)}{x\sin(xy) - \cos y}
\end{aligned}$$

$$\boxed{\frac{dy}{dx} = \frac{y\sin(xy)}{x\sin(xy) - \cos y}}$$

Example 9.4.
Find $\dfrac{dy}{dx}$ by implicit differentiation.

$$\tan\left(\frac{x}{y}\right) = x + y$$

Solution.

$$\begin{aligned}
\frac{d}{dx}\tan\left(\frac{x}{y}\right) &= \frac{d}{dx}(x + y) \\
\sec^2\left(\frac{x}{y}\right) \cdot \frac{d}{dx}\left(\frac{x}{y}\right) &= 1 + y'
\end{aligned}$$

$$\sec^2\left(\frac{x}{y}\right)\cdot\frac{1\cdot y - xy'}{y^2} = 1 + y'$$
$$(y - xy')\sec^2\left(\frac{x}{y}\right) = (1+y')y^2$$
$$\left[x\sec^2\left(\frac{x}{y}\right) + y^2\right]y' = y\sec^2\left(\frac{x}{y}\right) - y^2$$
$$y' = \frac{y\sec^2\left(\frac{x}{y}\right) - y^2}{x\sec^2\left(\frac{x}{y}\right) + y^2}$$

$$\boxed{\frac{dy}{dx} = y\left(\frac{\sec^2\left(\frac{x}{y}\right) - y}{x\sec^2\left(\frac{x}{y}\right) + y^2}\right)}$$

9.3. Tangent Line Problems

After finding y', we can evaluate it at a given point to find the slope of the tangent line. Then use the point-slope form:

$$y - y_1 = m(x - x_1).$$

Example 9.5.

Use implicit differentiation to find an equation of the tangent line to the curve at the given point.

$$x^2 - xy - y^2 = 1, \qquad (2, 1)$$

Solution. First, we want to find $\frac{dy}{dx}$.

$$\frac{d}{dx}(x^2 - xy - y^2) = \frac{d}{dx}(1)$$
$$2x - y - xy' - 2yy' = 0$$
$$y' = \frac{2x - y}{x + 2y}.$$

So, the slope of the tangent line at the point $(2,1)$ is

$$\begin{aligned}\left.\frac{dy}{dx}\right|_{(2,1)} &= \left.\frac{2x-y}{x+2y}\right|_{(2,1)}\\ &= \frac{2(2)-(1)}{(2)+2(1)}\\ &= \frac{3}{4}.\end{aligned}$$

Thus, an equation of the tangent line is

$$y-1=\frac{3}{4}(x-2).$$

Equivalently

$$\boxed{y=\frac{3}{4}x-\frac{1}{2}}$$

Example 9.6.

Use implicit differentiation to find an equation of the tangent line to the curve at the given point.

$$\sin(x+y)=2x-2y, \qquad (\pi,\pi)$$

Solution.

By implicit differentiating we can find $\frac{dy}{dx}$:

$$\begin{aligned}\frac{d}{dx}[\sin(x+y)] &= \frac{d}{dx}(2x-2y)\\ \cos(x+y)\cdot(1+y') &= 2-2y'\\ y'[2+\cos(x+y)] &= 2-\cos(x+y)\\ y' &= \frac{2-\cos(x+y)}{2+\cos(x+y)}.\end{aligned}$$

Thus, the slope of the tangent line at the point (π, pi) is

$$\begin{aligned}\left.\frac{dy}{dx}\right|_{(\pi,\pi)} &= \left.\frac{2-\cos(x+y)}{2+\cos(x+y)}\right|_{(\pi,\pi)}\\ &= \frac{2-\cos(\pi+\pi)}{2+\cos(\pi+\pi)}\end{aligned}$$

$$= \frac{2-\cos(2\pi)}{2+\cos(2\pi)}$$
$$= \frac{2-1}{2+1}$$
$$= \frac{1}{3}.$$

Therefore, an equation of the tangent line is

$$y - \pi = \frac{1}{3}(x - \pi).$$

Equivalently,

$$\boxed{y = \frac{1}{3}x + \frac{2}{3}\pi}$$

9.4. Finding Second Derivatives

Sometimes, we need to find the second derivative $y'' \left(= \dfrac{d^2y}{dx^2}\right)$.

To do this, first find y' using implicit differentiation, and then differentiate again.

Example 9.7.
Find y'' by implicit differentiation.

$$x^2 + 4y^2 = 4$$

Solution.

$$\frac{d}{dx}(x^2 + 4y^2) = \frac{d}{dx}(4)$$
$$2x + 8yy' = 0$$
$$x + 4yy' = 0 \tag{9.2}$$
$$y' = -\frac{x}{4y} \tag{9.3}$$

Now implicitly differentiate (2):

$$\frac{d}{dx}(x + 4yy') = \frac{d}{dx}(0)$$
$$1 + 4y'y' + 4yy'' = 0$$

$$y'' = -\frac{1+4(y')^2}{4y}$$

Then by (3),

$$\begin{aligned} y'' &= -\frac{1+4\left(-\frac{x}{4y}\right)^2}{4y} \\ &= -\frac{1+\frac{x^2}{4y^2}}{4y} \\ &= -\frac{4y^2+x^2}{16y^3}. \end{aligned}$$

$$\boxed{\frac{d^2y}{dx^2} = -\frac{4y^2+x^2}{16y^3}}$$

9.5. Functions Defined Implicitly

Sometimes, a function is defined by an equation involving the function itself. We can still differentiate both sides with respect to x.

Example 9.8.
If

$$f(x) + x^2[f(x)]^3 = 10$$

and

$$f(1) = 2,$$

find $f'(1)$.
Solution.

$$\begin{aligned} \frac{d}{dx}(f(x) + x^2[f(x)]^3) &= \frac{d}{dx}(10) \\ f'(x) + 2x[f(x)]^3 + 3x^2[f(x)]^2 f'(x) &= 0 \end{aligned}$$

This implies

$$f'(1) + 2 \cdot 1 \cdot [f(1)]^3 + 3 \cdot 1^2 \cdot [f(1)]^2 f'(1) = 0.$$

Because $f(1) = 2$ we then get

$$f'(1) + 2 \cdot 1 \cdot [f(1)]^3 + 3 \cdot 1^2 \cdot [f(1)]^2 f'(1) = 0$$

$$\begin{aligned}
f'(1) + 2 \cdot 2^3 + 3 \cdot 2^2 f'(1) &= 0 \\
f'(1) + 16 + 12f'(1) &= 0 \\
13f'(1) &= -16 \\
f'(1) &= -\frac{16}{13}.
\end{aligned}$$

$$\boxed{f'(1) = -\frac{16}{13}}$$

Example 9.9.
If

$$g(x) + x\sin(g(x)) = x^2,$$

find $g'(0)$.
Solution.

$$\begin{aligned}
\frac{d}{dx}[g(x) + x\sin(g(x))] &= 0 \\
g'(x) + \sin(g(x)) + x\cos(g(x)) \cdot g'(x) &= 0
\end{aligned}$$

Hence,

$$\begin{aligned}
g'(0) + \sin(g(0)) + 0 \cdot \cos(g(0)) \cdot g'(0) &= 0 \\
g'(0) + \sin(g(0)) &= 0 \\
g'(0) &= -\sin(g(0)).
\end{aligned}$$

From the equation

$$g(x) + x\sin(g(x)) = x^2$$

we get

$$g(0) = 0.$$

Thus,

$$g'(0) = -\sin(0).$$

So,

$$\boxed{g'(0) = 0}$$

Regard y as the Independent Variable

Sometimes we regard y as the independent variable and x as the dependent variable. In that case, we find

$$\frac{dx}{dy}.$$

Example 9.10.

Regard y as the independent variable and x as the dependent variable. Use implicit differentiation to find $\dfrac{dx}{dy}$.

$$x^4y^2 - x^3y + 2xy^3 = 0$$

Solution.

$$\begin{aligned}
\frac{d}{dy}(x^4y^2 - x^3y + 2xy^3) &= \frac{d}{dy}(0) \\
4x^3x'y^2 + x^4(2y) - 3x^2x'y - x^3(1) + 2x'y^3 + 2x(3y^2) &= 0 \\
(4x^3y^2 - 3x^2y + 2y^3)x' &= x^3 - 2x^4y - 6xy^2 \\
x' &= \frac{x^3 - 2x^4y - 6xy^2}{4x^3y^2 - 3x^2y + 2y^3}
\end{aligned}$$

Thus,

$$\boxed{\frac{dx}{dy} = \frac{x^3 - 2x^4y - 6xy^2}{4x^3y^2 - 3x^2y + 2y^3}}$$

Exercises

Find $\dfrac{dy}{dx}$ by implicit differentiation.

1.
$$x^2 + xy + y^2 = 7$$

2.
$$3x^2 - 2xy + y^2 = 5$$

3.
$$x^4 + x^2y^2 + y^3 = 5$$

4.
$$\frac{x^2}{x+y} = y^2 + 1$$

5.
$$\sqrt{x+y} = x^4 + y^4$$

6.
$$y\sin(x^2) = x\sin(y^2)$$

7.
$$\tan\left(\frac{x}{y}\right) = x + y$$

8.
$$\sin(xy) = \cos(x+y)$$

9.
$$\tan(x-y) = \frac{y}{1+x^2}$$

Regard y as the independent variable and find $\dfrac{dx}{dy}$.

10.
$$x^4y^2 - x^3y + 2xy^3 = 0$$

11.
$$y\sec x = x\tan y$$

Use implicit differentiation to find an equation of the tangent line to the curve at the given point.

12.
$$y\sin(2x) = x\cos(2y), \qquad \left(\frac{\pi}{2}, \frac{\pi}{4}\right)$$

13.
$$\sin(x+y) = 2x - 2y, \qquad (\pi, \pi)$$

14.
$$x^2 - xy - y^2 = 1, \qquad (2, 1)$$

15.
$$x^2 + 2xy + 4y^2 = 12, \qquad (2, 1)$$

16.
$$x^2 + y^2 = (2x^2 + 2y^2 - x)^2, \qquad \left(0, \frac{1}{2}\right)$$

Find y'' by implicit differentiation.

17.
$$x^2 + 4y^2 = 4$$

18.
$$x^2 + xy + y^2 = 3$$

19.
$$\sin y + \cos x = 1$$

20.
$$x^2 - xy + y^2 = 7$$

Find the indicated derivative.

21. If
$$f(x) + x^2[f(x)]^3 = 10$$
and
$$f(1) = 2,$$
find $f'(1)$.

22. If
$$g(x) + x\sin(g(x)) = x^2,$$
find $g'(0)$.

23. Let
$$F(x) = f(g(x)),$$
where
$$f(-2) = 8, \qquad f'(-2) = 4, \qquad g(5) = -2, \qquad g'(5) = 6.$$
Find $F'(5)$.

24. Let
$$r(x) = f(g(h(x))),$$
where
$$h(1) = 2, \qquad g(2) = 3, \qquad h'(1) = 4, \qquad g'(2) = 5, \qquad f'(3) = 6.$$
Find $r'(1)$.

Chapter 10

Rates of Change in the Natural and Social Sciences

10.1. Rates of Change in the Natural and Social Sciences

Where Does Calculus Appear?

One of the remarkable features of calculus is that the same mathematical ideas appear in many different disciplines. A derivative may represent

- velocity in physics,
- population growth in biology,
- marginal cost in economics,
- chemical reaction rates in chemistry,
- or the rate at which information changes in computer science.

Although the applications look different, the underlying mathematics is exactly the same.

Recall that if $y = f(x)$, then $\dfrac{dy}{dx}$ represents the rate of change of y with respect to x.

In many applications, x, more precisely, the independent variable, represents time, distance, population, number of workers, amount of product produced or another measurable quantity. The derivative measures how rapidly one quantity changes relative to another.

Velocity and Acceleration

Suppose the position of a particle at time t is given by

$$s = f(t).$$

Then:

$$v(t) = s'(t)$$

is called the **velocity** of the particle, and

$$a(t) = v'(t) = s''(t)$$

is called the **acceleration**.

Interpretation

- $v(t) > 0$: particle moves in the positive direction.
- $v(t) < 0$: particle moves in the negative direction.
- $v(t) = 0$: particle is at rest.
- $|v(t)|$: speed.

Speeding Up and Slowing Down

A particle is:

- **Speeding up** when velocity and acceleration have the same sign.
- **Slowing down** when velocity and acceleration have opposite signs.

v	a	Motion
$+$	$+$	Speeding up
$-$	$-$	Speeding up
$+$	$-$	Slowing down
$-$	$+$	Slowing down

10.2. Rates of Change in Biology

Derivatives are often used to describe how populations, masses, or concentrations change over time.

Examples:

- Growth rate of a bacterial population.
- Growth rate of a plant.
- Change in the mass of a laboratory animal.
- Change in blood glucose level after a meal.
- Rate at which a tumor grows.

If $P(t)$ denotes a population at time t, then

$$P'(t)$$

represents the instantaneous growth rate of the population.

10.3. Rates of Change in Chemistry and Physics

Examples:

- Rate at which a chemical concentration changes.
- Rate at which water drains from a tank.
- Rate at which temperature changes.
- Velocity and acceleration of moving objects.

If $V(t)$ denotes the amount of liquid in a tank, then

$$V'(t)$$

represents the rate at which the volume changes.

10.4. Rates of Change in Economics

If

$$C(x)$$

is the **cost function** (the cost of producing x items), then

$$C'(x)$$

is called the **marginal cost**. Similarly, for the **revenue function** $R(x)$ and the **profit function** $P(x) := R(x) - C(x)$ we have:

- $R'(x) =$ marginal revenue.
- $P'(x) =$ marginal profit.

Marginal quantities estimate how much the corresponding quantity changes when one additional unit is produced (at x).

Example 10.1.
A particle moves according to

$$s(t) = t^3 - 6t^2 + 9t.$$

(a) Find the velocity function.

(b) Find the acceleration function.

(c) Determine when the particle is at rest.

Solution.

(a) The velocity function is the derivative of the position function So,

$$v(t) = s'(t) = 3t^2 - 12t + 9.$$

(b) The acceleration function is the derivative of the velocity function:

$$a(t) = v'(t) = s''(t).$$

Thus,

$$a(t) = 6t - 12.$$

(c) The particle is at rest when

$$v(t) = 0.$$

So we solve

$$3t^2 - 12t + 9 = 0$$
$$3(t-1)(t-3) = 0$$

Therefore,

$$t = 1 \qquad \text{or} \qquad t = 3.$$

So, the particle is at rest at

$$\boxed{t = 1 \text{ and } t = 3.}$$

Example 10.2.

A particle moves according to

$$s(t) = t^3 - 9t^2 + 24t.$$

Determine the intervals where the particle is speeding up and where it is slowing down.

Solution. First find the velocity:

$$v(t) = s'(t) = 3t^2 - 18t + 24.$$

By factoring we get:

$$v(t) = 3(t^2 - 6t + 8) = 3(t-2)(t-4).$$

Now find the acceleration:

$$a(t) = v'(t) = s''(t) = 6t - 18 = 6(t-3).$$

A particle is speeding up when velocity and acceleration have the same sign.
A particle is slowing down when velocity and acceleration have opposite signs.
The important values (zeros of $v(t)$ and $a(t)$) are:

$$t = 2, \qquad t = 3, \qquad t = 4.$$

Interval	$v(t)$	$a(t)$	Conclusion
$(0, 2)$	$+$	$-$	slowing down
$(2, 3)$	$-$	$-$	speeding up
$(3, 4)$	$-$	$+$	slowing down
$(4, \infty)$	$+$	$+$	speeding up

Therefore, the particle is speeding up on

$$\boxed{(2,3) \cup (4,\infty)}$$

and slowing down on

$$\boxed{(0,2) \cup (3,4)}.$$

Example 10.3.

The population of bacteria in a laboratory culture after t hours is

$$P(t) = 500 + 40t + 3t^2.$$

Find:

(a) $P'(t)$,

(b) the growth rate after 5 hours,

(c) the growth rate after 10 hours.

Solution.

The population is given by

$$P(t) = 500 + 40t + 3t^2.$$

(a) Differentiate to obtain the growth rate:

$$P'(t) = 40 + 6t.$$

(b) After 5 hours,

$$P'(5) = 40 + 6(5) = 40 + 30 = 70.$$

Thus, the growth rate after 5 hours is

$$\boxed{\text{70 bacteria per hour.}}$$

(c) After 10 hours,

$$P'(10) = 40 + 6(10) = 40 + 60 = 100.$$

Thus, the growth rate after 10 hours is

$$\boxed{\text{100 bacteria per hour.}}$$

Example 10.4.

Water drains from a tank according to

$$V(t) = 1000\left(1 - \frac{t}{20}\right)^2, \qquad 0 \leq t \leq 20.$$

Here, $V(t)$ is the volume of water remaining in the tank after t minutes. Find the rate at which water is leaving the tank after 5 minutes.

Solution.

The volume of water remaining in the tank is

$$V(t) = 1000\left(1 - \frac{t}{20}\right)^2.$$

By differentiating using the Chain Rule:

$$V'(t) = 1000 \cdot 2\left(1 - \frac{t}{20}\right)\left(-\frac{1}{20}\right).$$

By simplifying we get:

$$V'(t) = -100\left(1 - \frac{t}{20}\right).$$

Now evaluate at $t = 5$:

$$V'(5) = -100\left(1 - \frac{5}{20}\right) = -100\left(\frac{3}{4}\right) = -75.$$

The negative sign indicates that the volume of water is decreasing. Therefore, water is leaving the tank at a rate of

$$\boxed{\text{75 liters per minute.}}$$

Example 10.5.

The cost (in dollars) of producing x units of a product is

$$C(x) = 1000 + 5x + 0.02x^2.$$

(a) Find the marginal cost function.

(b) Find $C'(100)$.

(c) Interpret the meaning of $C'(100)$.

Solution.

(a)

$$\begin{aligned} C'(x) &= \frac{d}{dx}\left(1000 + 5x + 0.02x^2\right) \\ &= 0 + 5(1) + 0.02(2x^{2-1}) \\ &= 0.04x + 5. \end{aligned}$$

(b) $C'(100) = 0.04(100) + 5 = 4 + 5 = 9.$

(c) Since $C'(100) = 9$, when the company is producing 100 units, the cost is increasing at a rate of approximately

$$\boxed{\text{\$9 per additional unit}}.$$

In practical terms, the cost of producing the 101^{st} unit is approximately \$9.

Equivalently, increasing production from 100 units to 101 units is predicted to increase the total cost by about \$9.

Example 10.6.

A company manufactures and sells x units of a product. The cost (in dollars) of producing x units is

$$C(x) = 5000 + 12x + 0.04x^2,$$

and the profit (in dollars) is

$$P(x) = 18x - 0.01x^2.$$

(a) Find the revenue function $R(x)$.

(b) Find the marginal revenue function.

(c) Find the marginal revenue when $x = 100$.

(d) Interpret the meaning of $R'(100)$.

Solution.
Recall that

$$\text{Profit} = \text{Revenue} - \text{Cost}.$$

Hence,

$$R(x) = P(x) + C(x).$$

(a) The revenue function is

$$\begin{aligned} R(x) &= P(x) + C(x) \\ &= \left(18x - 0.01x^2\right) + \left(5000 + 12x + 0.04x^2\right) \\ &= 5000 + 30x + 0.03x^2. \end{aligned}$$

Thus,

$$\boxed{R(x) = 5000 + 30x + 0.03x^2.}$$

(b) The marginal revenue function is the derivative of the revenue function:

$$R'(x) = 30 + 0.06x.$$

Thus,

$$\boxed{R'(x) = 30 + 0.06x.}$$

(c) When $x = 100$,

$$R'(100) = 30 + 0.06(100) = 30 + 6 = 36.$$

Therefore,

$$\boxed{R'(100) = 36.}$$

(d) The value

$$R'(100) = 36$$

means that when the company is producing and selling approximately 100 units, the revenue is increasing at a rate of about

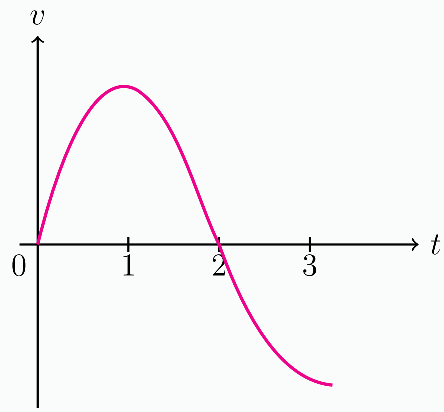
$$\boxed{\$36}$$

for each additional unit sold.

Equivalently, producing and selling one more unit beyond 100 units is expected to increase the revenue by approximately \$36.

Example 10.7.

The graph below shows the velocity function $v(t)$ of a particle moving along a line, where t is measured in seconds.



When is the particle speeding up? When is it slowing down? Explain.

Solution.

Recall:

A particle is speeding up when $v(t)$ and $a(t)$ have the same sign.

A particle is slowing down when $v(t)$ and $a(t)$ have opposite signs.

Also,

$$a(t) = v'(t),$$

so the acceleration is the slope of the velocity graph. From the graph:

$$v(t) > 0$$

on approximately

$$0 < t < 2,$$

and

$$v(t) < 0$$

for approximately

$$t > 2.$$

The velocity graph is increasing on approximately

$$0 < t < 1,$$

so

$$a(t) > 0$$

there.
The velocity graph is decreasing on approximately

$$t > 1,$$

so

$$a(t) < 0$$

there.
Therefore:

$$\boxed{\text{The particle is speeding up on } (0,1) \text{ and } (2,3).}$$

$$\boxed{\text{The particle is slowing down on } (1,2).}$$

Explanation:

On $(0,1)$, the velocity is positive and increasing, so

$$v(t) > 0 \qquad \text{and} \qquad a(t) > 0.$$

Thus the particle is speeding up.

On $(1,2)$, the velocity is positive but decreasing, so

$$v(t) > 0 \qquad \text{and} \qquad a(t) < 0.$$

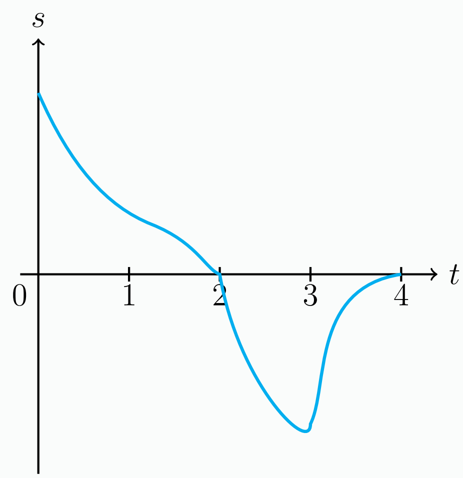
Thus the particle is slowing down.
On $(2,3)$, the velocity is negative and decreasing, so

$$v(t) < 0 \qquad \text{and} \qquad a(t) < 0.$$

Thus the particle is speeding up.

Example 10.8.

The graph below shows the position function $s(t)$ of a particle moving along a line, where t is measured in seconds.



When is the particle speeding up? When is it slowing down? Explain.

Solution.

Since the graph shows the position function $s(t)$, we use:

$$v(t) = s'(t) \qquad \text{and} \qquad a(t) = s''(t).$$

So:

- $v(t) > 0$ when $s(t)$ is increasing.
- $v(t) < 0$ when $s(t)$ is decreasing.
- $a(t) > 0$ when $s(t)$ is concave up.
- $a(t) < 0$ when $s(t)$ is concave down.

A particle is speeding up when velocity and acceleration have the same sign.
A particle is slowing down when velocity and acceleration have opposite signs.
From the graph, $s(t)$ is decreasing on approximately

$$(0, 3),$$

so

$$v(t) < 0$$

on this interval.

The graph appears concave up on approximately

$$(0, 2),$$

so

$$a(t) > 0$$

there.
Thus, on

$$(0, 2),$$

we have

$$v(t) < 0 \qquad \text{and} \qquad a(t) > 0.$$

Therefore, the particle is slowing down on

$$\boxed{(0, 2)}.$$

From approximately

$$(2, 3),$$

the graph is decreasing and concave down, so

$$v(t) < 0 \qquad \text{and} \qquad a(t) < 0.$$

Therefore, the particle is speeding up on

$$\boxed{(2, 3)}.$$

After about $t = 3$, the graph is increasing and concave up, so

$$v(t) > 0 \qquad \text{and} \qquad a(t) > 0.$$

Therefore, the particle is speeding up on approximately

$$\boxed{(3, 4)}.$$

Hence:

$$\boxed{\text{Slowing down on approximately } (0, 2).}$$

$$\boxed{\text{Speeding up on approximately } (2, 3) \cup (3, 4).}$$

Exercises

1. A particle moves according to
$$s(t) = t^3 - 12t.$$
Find its velocity and acceleration.

2. A particle moves according to
$$s(t) = t^3 - 9t^2 + 24t.$$
Determine when it is at rest.

3. For the particle in Problem 2, determine when it is moving in the positive direction.

4. For the particle in Problem 2, determine when it is speeding up and when it is slowing down.

5. The position function of a particle is
$$s(t) = 2t^3 - 15t^2 + 24t.$$
Find the total distance traveled during the first 5 seconds.

6. The population of fish in a pond is
$$P(t) = 1000 + 60t + 2t^2.$$
Find the growth rate after 10 years.

7. The mass of a laboratory animal is
$$m(t) = 2 + 0.5t + 0.03t^2.$$
Find the rate at which the mass increases after 8 weeks.

8. The water drains from a tank according to
$$V(t) = 2000\left(1 - \frac{t}{40}\right)^2.$$
Find the rate at which the water drains after 10 minutes.

9. A ball is thrown upward with height
$$s(t) = 96t - 16t^2.$$

Find its maximum height.

10. A ball is thrown upward with height

$$s(t) = 64t - 16t^2.$$

Find its velocity when it is 48 feet above the ground.

11. The cost function is

$$C(x) = 1500 + 4x + 0.01x^2.$$

Find the marginal cost function.

12. Find the marginal cost when $x = 200$.

13. The revenue function is

$$R(x) = 50x - 0.05x^2.$$

Find the marginal revenue function.

14. The profit function is

$$P(x) = 500x - 0.2x^2 - 10000.$$

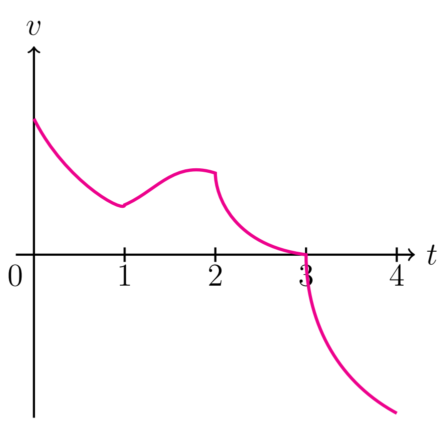
Find the marginal profit function.

15. Explain in your own words the meaning of:

$$P'(t) = 120.$$

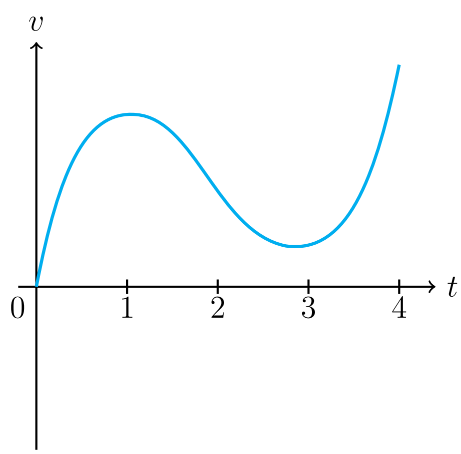
when $P(t)$ represents a population.

16. The graph below shows the velocity function $v(t)$ of a particle moving along a line, where t is measured in seconds.



When is the particle speeding up? When is it slowing down? Explain.

17. The graph below shows the velocity function $v(t)$ of a particle moving along a line, where t is measured in seconds.



When is the particle speeding up? When is it slowing down? Explain.

Chapter 11

Related Rates

11.1. Related Rates

Definition

Related Rates

In many applications, several quantities change with respect to time and are related by an equation.

The goal of a **related rates problem** is to determine the rate at which one quantity is changing by using information about the rates of change of other related quantities.

If a quantity depends on time, then its derivative with respect to time represents its rate of change.

$$\frac{dx}{dt}, \qquad \frac{dy}{dt}, \qquad \frac{dV}{dt}, \qquad \frac{dA}{dt}$$

are examples of related rates.

Many related rates problems involve geometry.

Typical quantities include:

- lengths
- areas
- volumes
- angles
- distances
- heights
- velocities
- populations
- costs and revenues

The key idea is to:

1. write an equation relating the variables,
2. differentiate implicitly with respect to time,
3. substitute the given information,
4. solve for the unknown rate.

Warning

Problem-Solving Strategy

1. Read the problem carefully.
2. Draw a diagram whenever possible.
3. Introduce notation and identify all quantities that vary with time.
4. Determine which rate is given and which rate is required.
5. Write an equation relating the variables.
6. Differentiate both sides with respect to time.
7. Substitute the numerical information only after differentiating.
8. Solve for the unknown rate.

Warning

Common Mistake
A very common mistake is to substitute numerical values before differentiating.
For example, if the radius of a sphere is currently $r = 5$, do *not* replace r by 5 before taking derivatives.
Differentiate first:

$$V = \frac{4}{3}\pi r^3$$

$$\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$$

and only then substitute the numerical values.

11.2. Examples

Example 11.1.

A square is expanding so that each side increases at a rate of 6 cm/s.
Find the rate at which the area is increasing when the area is 16 cm^2.

Solution.

Let

$$\begin{aligned} x &:= \text{length of a side of the square} \\ A \equiv A(x) &:= \text{area of the square} \\ t &:= \text{time} \end{aligned}$$

Then,

$$A(x) = x^2.$$

By differentiating with respect to time t we get

$$\frac{d}{dt}A(x) = 2xx'.$$

Because $x' = 6$, we then get

$$\frac{d}{dt}A(x) = 12x.$$

Notice that when

$$A(x) = 16,$$

we have

$$x = 4.$$

Thus,

$$\begin{aligned} \frac{d}{dt}A(x)\bigg|_{x=4} &= 12(4) \\ &= 48. \end{aligned}$$

$$\boxed{\text{The are is increasing at 48 cm}^2/s}$$

Example 11.2.
The length of a rectangle is increasing at a rate of 8 cm/s and its width is increasing at a rate of 3 cm/s.
How fast is the area increasing when the length is 20 cm and the width is 10 cm?

Solution.
Let

$$L = \text{length}, \qquad W = \text{width}, \qquad A = \text{area}.$$

The area of a rectangle is

$$A = LW.$$

Differentiate both sides with respect to time t:

$$\frac{dA}{dt} = L\frac{dW}{dt} + W\frac{dL}{dt}.$$

We are given

$$\frac{dL}{dt} = 8 \text{ cm/s}, \qquad \frac{dW}{dt} = 3 \text{ cm/s},$$

and at the instant of interest,

$$L = 20 \text{ cm}, \qquad W = 10 \text{ cm}.$$

Substitute these values into the related rates equation:

$$\frac{dA}{dt} = 20(3) + 10(8) = 60 + 80 = 140.$$

Therefore, the area is increasing at a rate of

$$\boxed{\frac{dA}{dt} = 140 \text{ cm}^2/\text{s}.}$$

Example 11.3.
A cylindrical tank of radius 5 m is being filled with water at a rate of

$$3 \text{ m}^3/\text{min}.$$

How fast is the water level rising?
Solution.

Let

$$V = \text{volume of water} \qquad \text{and} \qquad h = \text{height of the water.}$$

The volume of a cylinder is

$$V = \pi r^2 h.$$

Since the radius is constant,

$$r = 5 \text{ m},$$

we have

$$V = 25\pi h.$$

Differentiate both sides with respect to time t:

$$\frac{dV}{dt} = 25\pi \frac{dh}{dt}.$$

We are given

$$\frac{dV}{dt} = 3 \text{ m}^3/\text{min}.$$

By substituting this value into the equation we get

$$3 = 25\pi \frac{dh}{dt}.$$

By solving for $\frac{dh}{dt}$ we get

$$\frac{dh}{dt} = \frac{3}{25\pi}.$$

Therefore, the water level is rising at a rate of

$$\boxed{\frac{3}{25\pi} \text{ m/min.}}$$

Example 11.4.

The radius of a sphere is increasing at a rate of 4 mm/s.
How fast is the volume increasing when the diameter is 80 mm?

Solution.

Let

$$r = \text{radius of the sphere} \qquad \text{and} \qquad V = \text{volume of the sphere.}$$

The volume of a sphere is

$$V = \frac{4}{3}\pi r^3.$$

Differentiate both sides with respect to time t:

$$\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}.$$

We are given

$$\frac{dr}{dt} = 4 \text{ mm/s}.$$

The diameter is 80 mm, so the radius is

$$r = 40 \text{ mm}.$$

Substitute these values into the related rates equation we get

$$\frac{dV}{dt} = 4\pi(40)^2(4).$$

By simplifying we get

$$\frac{dV}{dt} = 4\pi(1600)(4) = 25600\pi.$$

Therefore, the volume is increasing at a rate of

$$\boxed{25,600\pi \text{ mm}^3/\text{s}.}$$

Example 11.5.

The radius of a spherical balloon is increasing at a rate of 2 cm/min.
At what rate is the surface area increasing when the radius is 8 cm?

Solution.

Let

$$r = \text{radius of the balloon} \qquad \text{and} \qquad S = \text{surface area of the balloon.}$$

The surface area of a sphere is

$$S = 4\pi r^2.$$

Differentiate both sides with respect to time t:

$$\frac{dS}{dt} = 8\pi r \frac{dr}{dt}.$$

We are given

$$\frac{dr}{dt} = 2 \text{ cm/min},$$

and we want the rate when
$$r = 8 \text{ cm}.$$
By substituting these values into the related rates equation we get
$$\frac{dS}{dt} = 8\pi(8)(2) = 128\pi.$$
Therefore, the surface area is increasing at a rate of
$$\boxed{128\pi \text{ cm}^2/\text{min}.}$$

Example 11.6.
Suppose
$$y = \sqrt{2x+1},$$
where both x and y are functions of time. If $\frac{dx}{dt} = 3$, find $\frac{dy}{dt}$ when $x = 4$.

Solution.
By differentiating both sides of
$$y = \sqrt{2x+1}$$
with respect to time t, we get
$$\frac{dy}{dt} = \frac{1}{2\sqrt{2x+1}} \cdot \frac{d}{dt}(2x+1).$$
By using the Chain Rule we get
$$\frac{dy}{dt} = \frac{1}{2\sqrt{2x+1}} \cdot 2 \cdot \frac{dx}{dt} = \frac{1}{\sqrt{2x+1}}\frac{dx}{dt}.$$
We are given
$$\frac{dx}{dt} = 3 \qquad \text{and} \qquad x = 4.$$
Thus, by substituting these values we get
$$\frac{dy}{dt} = \frac{3}{\sqrt{2(4)+1}} = \frac{3}{\sqrt{9}} = \frac{3}{3} = 1.$$
Therefore,
$$\boxed{\frac{dy}{dt} = 1.}$$

Example 11.7.

Suppose

$$4x^2 + 9y^2 = 36,$$

where x and y are functions of time. If $\frac{dy}{dt} = \frac{1}{3}$, find $\frac{dx}{dt}$ when $x = 2$ and $y = \frac{2\sqrt{5}}{3}$.

Solution.

By differentiating both sides of

$$4x^2 + 9y^2 = 36$$

with respect to time t we get

$$8x\frac{dx}{dt} + 18y\frac{dy}{dt} = 0.$$

By solving for $\frac{dx}{dt}$ we get

$$8x\frac{dx}{dt} = -18y\frac{dy}{dt}.$$

Thus,

$$\frac{dx}{dt} = -\frac{18y}{8x}\frac{dy}{dt} = -\frac{9y}{4x}\frac{dy}{dt}.$$

We are given

$$x = 2, \qquad y = \frac{2\sqrt{5}}{3}, \qquad \frac{dy}{dt} = \frac{1}{3}.$$

By substituting these values we get

$$\frac{dx}{dt} = -\frac{9}{4(2)}\left(\frac{2\sqrt{5}}{3}\right)\left(\frac{1}{3}\right).$$

By simplifying

$$\frac{dx}{dt} = -\frac{18\sqrt{5}}{72} = -\frac{\sqrt{5}}{4}.$$

Therefore,

$$\boxed{\frac{dx}{dt} = -\frac{\sqrt{5}}{4}.}$$

Example 11.8.

If

$$x^2 + y^2 + z^2 = 9,$$

and

$$\frac{dx}{dt} = 5, \qquad \frac{dy}{dt} = 4,$$

find

$$\frac{dz}{dt} \quad \text{when} \quad (x, y, z) \equiv (2, 2, 1).$$

Solution.

By differentiating both sides of

$$x^2 + y^2 + z^2 = 9$$

with respect to time t we get

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} + 2z\frac{dz}{dt} = 0.$$

By solving for $\frac{dz}{dt}$ we get

$$2z\frac{dz}{dt} = -2x\frac{dx}{dt} - 2y\frac{dy}{dt}.$$

Thus,

$$\frac{dz}{dt} = -\frac{x\frac{dx}{dt} + y\frac{dy}{dt}}{z}.$$

We are given

$$x = 2, \qquad y = 2, \qquad z = 1,$$

and

$$\frac{dx}{dt} = 5, \qquad \frac{dy}{dt} = 4.$$

Hence, by substituting these values we get

$$\frac{dz}{dt} = -\frac{2(5) + 2(4)}{1} = -(10 + 8) = -18.$$

Therefore,

$$\boxed{\frac{dz}{dt} = -18.}$$

Example 11.9.

A street light is mounted at the top of a 15-ft pole.
A man who is 6 ft tall walks away from the pole at 5 ft/s.
How fast is the tip of his shadow moving when he is 40 ft from the pole?

Solution.
Let

$$x = \text{distance from the pole to the man,}$$

and

$$y = \text{length of the man's shadow.}$$

Then the distance from the pole to the tip of the shadow is

$$x + y.$$

We are given

$$\frac{dx}{dt} = 5 \text{ ft/s,}$$

and we want to find

$$\frac{d}{dt}(x + y).$$

Using similar triangles we get

$$\frac{15}{x + y} = \frac{6}{y}.$$

By simplifying:

$$15y = 6x + 6y.$$

Thus,

$$y = \frac{2}{3}x.$$

Differentiate both sides with respect to time we get

$$\frac{dy}{dt} = \frac{2}{3}\frac{dx}{dt}.$$

Now substitute $\dfrac{dx}{dt} = 5$:

$$\frac{dy}{dt} = \frac{2}{3}(5) = \frac{10}{3} \text{ ft/s.}$$

Therefore,

$$\frac{d}{dt}(x+y) = \frac{dx}{dt} + \frac{dy}{dt} = 5 + \frac{10}{3} = \frac{25}{3} \text{ ft/s.}$$

Notice that this rate is independent of the value of x, so in particular, when the man is 40 ft from the pole,

$$\boxed{\frac{25}{3} \text{ ft/s.}}$$

Example 11.10.

At noon, Ship A is 150 km west of Ship B.
Ship A travels east at 35 km/h and Ship B travels north at 25 km/h.
How fast is the distance between the ships changing at 4:00 PM?

Solution.
Let

$$\begin{aligned} x &:= \text{horizontal distance between the ships} \\ y &:= \text{vertical distance between the ships} \\ z &:= \text{distance between the ships.} \end{aligned}$$

Then,

$$z^2 = x^2 + y^2.$$

At noon, the ships are 150 km apart horizontally. Since Ship A travels east at 35 km/h, the horizontal distance decreases at

$$\frac{dx}{dt} = -35 \text{ km/h.}$$

Ship B travels north at 25 km/h, so

$$\frac{dy}{dt} = 25 \text{ km/h.}$$

After 4 hours,

$$x = 150 - 35(4) = 10 \text{ km},$$

$$y = 25(4) = 100 \text{ km.}$$

Thus,

$$z = \sqrt{10^2 + 100^2} = \sqrt{10100} = 10\sqrt{101} \text{ km.}$$

By differentiating

$$z^2 = x^2 + y^2$$

with respect to time we get

$$2z\frac{dz}{dt} = 2x\frac{dx}{dt} + 2y\frac{dy}{dt}.$$

Therefore,

$$\frac{dz}{dt} = \frac{x\frac{dx}{dt} + y\frac{dy}{dt}}{z}.$$

By substituting the known values we get

$$\frac{dz}{dt} = \frac{10(-35) + 100(25)}{10\sqrt{101}} = \frac{2150}{10\sqrt{101}} = \frac{215}{\sqrt{101}}.$$

Hence, the distance between the ships is changing at

$$\boxed{\frac{215}{\sqrt{101}} \text{ km/h}}$$

or approximately

$$\boxed{21.39 \text{ km/h.}}$$

Exercises

1. A square has a side length increasing at a rate of 4 cm/s. How fast is the area increasing when the side length is 10 cm?

2. The radius of a circle is increasing at a rate of 3 cm/min. How fast is the area increasing when the radius is 5 cm?

3. The radius of a sphere is increasing at a rate of 2 mm/s. How fast is the volume increasing when the radius is 10 mm?

4. A cylindrical tank of radius 6 m is being filled with water at a rate of

$$5 \text{ m}^3/\text{min}.$$

   How fast is the water level rising?

5. Water is draining from a tank according to

$$V = 4000\left(1 - \frac{t}{40}\right)^2, \qquad 0 \le t \le 40.$$

   Find the rate at which water is leaving the tank when

   (a) $t = 10$,
   (b) $t = 20$,
   (c) $t = 30$.

6. Suppose

$$y = \sqrt{3x + 1},$$

   where x and y are functions of time.

   If

$$\frac{dx}{dt} = 4,$$

   find

$$\frac{dy}{dt}$$

   when $x = 5$.

7. Suppose

$$x^2 + y^2 = 100,$$

where x and y are functions of time.

If

$$\frac{dx}{dt} = 3,$$

find

$$\frac{dy}{dt}$$

when

$$(x, y) = (6, 8).$$

8. Suppose

$$4x^2 + 9y^2 = 36,$$

where x and y depend on time.

If

$$\frac{dy}{dt} = \frac{1}{2},$$

find

$$\frac{dx}{dt}$$

when

$$x = \frac{3}{2}, \qquad y = 1.$$

9. If

$$x^2 + y^2 + z^2 = 25,$$

and

$$\frac{dx}{dt} = 2, \qquad \frac{dy}{dt} = 1,$$

find

$$\frac{dz}{dt}$$

at the point

$$(3, 4, 0).$$

10. A ladder 13 ft long leans against a wall.

    The bottom slides away from the wall at a rate of 2 ft/s.

    How fast is the top of the ladder moving when the bottom is 5 ft from the wall?

11. A street light is mounted at the top of a 12-ft pole.

    A person 5 ft tall walks away from the pole at 4 ft/s.

    How fast is the tip of the shadow moving when the person is 20 ft from the pole?

12. At noon, Ship A is 100 km west of Ship B.

    Ship A travels east at 20 km/h while Ship B travels north at 15 km/h.

    How fast is the distance between the ships changing after 3 hours?

13. A kite is flying at a height of 80 ft.

    The kite moves horizontally at 6 ft/s.

    At what rate is the length of the string increasing when 100 ft of string has been let out?

14. A kite is flying 120 ft above the ground and moves horizontally at 8 ft/s.

    At what rate is the angle between the string and the horizontal changing when 200 ft of string has been let out?

15. The side length of an equilateral triangle is increasing at a rate of 5 cm/min.

    How fast is the area increasing when the side length is 24 cm?

16. A spherical balloon is expanding so that its radius increases at a rate of 1.5 cm/min.

    How fast is the surface area increasing when the radius is 10 cm?

17. A particle moves along the curve

    $$y = 2\sin\left(\frac{\pi x}{2}\right).$$

    The x-coordinate increases at a rate of 2 cm/s.

    Find

    $$\frac{dy}{dt}$$

    when

    $$x = \frac{1}{3}.$$

18. The population of a bacterial culture is modeled by

    $$P(t) = 500 + 40t - 0.5t^2.$$

    Find the rate at which the population is changing after

    (a) 5 hours,

    (b) 20 hours.

19. The mass (in grams) of a growing plant is modeled by

$$M(t) = 100 + 12t + 0.2t^2.$$

Find the growth rate after 10 days.

20. The cost (in dollars) of producing x units is

$$C(x) = 1000 + 4x + 0.01x^2.$$

Find the marginal cost function and evaluate it at $x = 200$.

Chapter 12

Linear Approximations and Differentials

12.1. Linear Approximations and Differentials

Idea:
A differentiable function is very close to its tangent line near the point of tangency.
Therefore, we can use the tangent line to approximate the value of a function near a point.
This process is called a **linear approximation** or **linearization**.

Suppose f is differentiable at $x = a$.

The tangent line at $(a, f(a))$ has the equation

$$y = f(a) + f'(a)(x - a).$$

Therefore,

$$f(x) \approx f(a) + f'(a)(x - a)$$

for values of x close to a.

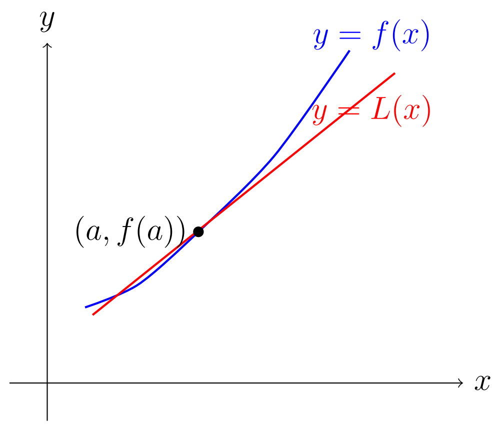
Formula

$$L(x) = f(a) + f'(a)(x - a)$$

is called the **linearization** of f at $x = a$.

12.1.1 Geometric Interpretation

The graph of a differentiable function looks more and more like its tangent line when we zoom in near the point of tangency.



Near $x = a$, we may assume $f(x) \approx L(x)$.

12.1.2 Applications of Linear Approximations

Linear approximations are useful when nearby function values are difficult to compute exactly.

Common applications include:

- Approximating square roots.
- Approximating cube roots.
- Approximating trigonometric values.
- Estimating values of complicated functions.
- Estimating measurement errors.

Example 12.1.
Find the linearization of $f(x) = \sqrt{x}$ at $a = 4$.

Solution.
The linearization of a function $f(x)$ at $x = a$ is

$$L(x) = f(a) + f'(a)(x - a).$$

Here,

$$f(x) = \sqrt{x} = x^{1/2},$$

thus,

$$f'(x) = \frac{1}{2\sqrt{x}}.$$

Evaluate the function and its derivative at $a = 4$:

$$f(4) = 2,$$

and

$$f'(4) = \frac{1}{2\sqrt{4}} = \frac{1}{4}.$$

Therefore,

$$L(x) = 2 + \frac{1}{4}(x - 4).$$

Simplifying,

$$L(x) = 2 + \frac{x}{4} - 1 = 1 + \frac{x}{4}.$$

Hence, the linearization of $f(x) = \sqrt{x}$ at $a = 4$ is

$$\boxed{L(x) = 2 + \frac{1}{4}(x - 4) = 1 + \frac{x}{4}.}$$

Example 12.2.

Use a linear approximation to estimate

$$\sqrt{4.1}.$$

Solution.

From the previous problem, the linearization of

$$f(x) = \sqrt{x}$$

at $a = 4$ is

$$L(x) = 2 + \frac{1}{4}(x - 4).$$

To estimate $\sqrt{4.1}$, substitute $x = 4.1$ into the linearization:

$$\sqrt{4.1} \approx L(4.1) = 2 + \frac{1}{4}(4.1 - 4).$$

By simplifying we get:

$$L(4.1) = 2 + \frac{1}{4}(0.1) = 2 + 0.025 = 2.025.$$

Therefore,

$$\boxed{\sqrt{4.1} \approx 2.025.}$$

Example 12.3.

Use a linear approximation to estimate

$$(1.98)^4.$$

Solution.

Let

$$f(x) = x^4.$$

Choose the point

$$a = 2,$$

since 1.98 is close to 2.

The linearization of f at $a = 2$ is

$$L(x) = f(2) + f'(2)(x - 2).$$

First compute

$$f(2) = 2^4 = 16,$$

and

$$f'(x) = 4x^3,$$

so

$$f'(2) = 4(2^3) = 32.$$

Therefore,

$$L(x) = 16 + 32(x - 2).$$

Now substitute $x = 1.98$:

$$(1.98)^4 \approx L(1.98) = 16 + 32(1.98 - 2).$$

By simplifying we get

$$L(1.98) = 16 + 32(-0.02) = 16 - 0.64 = 15.36.$$

Hence,

$$\boxed{(1.98)^4 \approx 15.36.}$$

12.2. Differentials

Suppose

$$y = f(x).$$

The **differential of x** is denoted by dx.

The **differential of y** is defined by

$$dy = f'(x)\,dx.$$

For small changes in x,

$$\Delta y = f(x + \Delta x) - f(x)$$

is approximately equal to

$$dy.$$

Formula

$$dy = f'(x)\,dx$$

and when Δx is small,

$$\Delta y \approx dy.$$

Example 12.4.
Find the differential of

$$y = x^3 - 2x + 5.$$

Solution.
By the power rule for differentiation we get

$$\frac{dy}{dx} = 3x^2 - 2.$$

Thus,

$$\boxed{dy = (3x^2 - 2)dx}$$

Example 12.5.
Find the differential of

$$y = \sqrt{1 + x^2}.$$

Solution.
We have

$$\frac{dy}{dx} = \frac{x}{\sqrt{1 + x^2}}.$$

Thus,

$$\boxed{dy = \frac{xdx}{\sqrt{1 + x^2}}}$$

12.3. Relative Error and Percentage Error

Differentials can be used to estimate errors in measurements.

If Q represents a measured quantity, then dQ approximates the error in Q.

The relative error is

$$\frac{|dQ|}{|Q|}.$$

The percentage error is

$$\frac{|dQ|}{|Q|} \times 100\%.$$

Formula

$$\text{Relative Error} = \frac{|dQ|}{|Q|}$$

$$\text{Percentage Error} = \frac{|dQ|}{|Q|} \times 100\%.$$

Example 12.6.

The radius of a sphere is measured as

$$r = 10 \text{ cm}$$

with a possible error

$$dr = 0.05 \text{ cm}.$$

Use differentials to estimate the maximum error in the volume.

Solution.

The volume V of the sphere is given by

$$V = \frac{4}{3}\pi r^3.$$

By differentiating with respect to r we get

$$\frac{dV}{dr} = \frac{8}{3}\pi r^2.$$

The maximum error in the volume is approximated by

$$|dV|.$$

By substituting

$$r = 10 \text{ cm}, \qquad dr = 0.05 \text{ cm}$$

we get

$$dV = 4\pi(100)(0.05) = 20\pi.$$

Therefore, the maximum error in the volume is approximately

$$\boxed{20\pi \text{ cm}^3 \approx 62.83 \text{ cm}^3}$$

Example 12.7.
A circular disk has radius
$$r = 20 \text{ cm}$$
with possible measurement error
$$dr = 0.1 \text{ cm}.$$
Estimate:

(a) the maximum error in the area,

(b) the relative error,

(c) the percentage error.

Solution.

(a) The area A of a circle is given by
$$A = \pi r^2.$$
By differentiating with respect to r, we can obtain the differential
$$dA = 2\pi r\, dr.$$
The maximum error in the area is approximated by
$$|dA|.$$
Since
$$r = 20 \text{ cm} \qquad \text{and} \qquad dr = 0.1 \text{ cm},$$
we have
$$dA = 2\pi(20)(0.1) = 4\pi.$$
Therefore,
$$\boxed{\text{Maximum error in the area} = 4\pi \text{ cm}^2 \approx 12.57 \text{ cm}^2}$$

(b) The relative error is approximately

$$\frac{|dA|}{A}.$$

Because

$$A = \pi(20)^2 = 400\pi,$$

we get

$$\frac{|dA|}{A} = \frac{4\pi}{400\pi} = \frac{1}{100} = 0.01.$$

Thus,

$$\boxed{\text{Relative error } = 0.01.}$$

(c) The percentage error is

$$0.01 \times 100\% = 1\%.$$

Hence,

$$\boxed{\text{Percentage error } \approx 1\%.}$$

Exercises

1. Find the linearization of
$$f(x) = x^3 - 2x + 1$$
at $a = 1$.

2. Find the linearization of
$$f(x) = \sqrt{x}$$
at $a = 9$.

3. Find the linearization of
$$f(x) = \sin x$$
at
$$a = \frac{\pi}{6}.$$

4. Use a linear approximation to estimate
$$\sqrt{15.8}.$$

5. Use a linear approximation to estimate
$$(2.03)^3.$$

6. Use a linear approximation to estimate
$$\sqrt[3]{26}.$$

7. Use a linear approximation to estimate
$$\cos(0.08).$$

8. Find the differential of
$$y = x^4 + 3x^2.$$

9. Find the differential of
$$y = \frac{1}{x}.$$

10. Find the differential of
$$y = \sqrt{4x + 1}.$$

11. Find the differential of
$$y = \sin(x^2).$$

12. The radius of a sphere is measured as 15 cm with a possible error 0.2 cm. Use differentials to estimate the maximum error in the volume.

13. The side length of a cube is measured as 8 cm with a possible error 0.05 cm. Estimate the maximum error in the volume.

14. A circular disk has a radius 24 cm with a possible error 0.2 cm. Estimate the relative error and percentage error in the area.

15. Suppose
$$g(2) = -4 \qquad \text{and} \qquad g'(2) = 3.$$
Use a linear approximation to estimate
$$g(2.05)$$
and
$$g(1.98).$$

Chapter 13

Maximum and Minimum Values

13.1. Maximum and Minimum Values

Brief Description

In many applications, we want to determine the largest or smallest value of a function. Such values are called *extreme values*. In this section, we introduce absolute and local extrema, the Extreme Value Theorem, critical numbers, Fermat's Theorem, and the Closed Interval Method for finding absolute extrema.

13.2. Absolute and Local Extrema

Absolute Extrema

Let f be defined on a domain D and let $c \in D$.

- $f(c)$ is an **absolute maximum value** of f on D if

$$f(c) \geq f(x) \qquad \text{for all } x \in D.$$

- $f(c)$ is an **absolute minimum value** of f on D if

$$f(c) \leq f(x) \qquad \text{for all } x \in D.$$

An absolute maximum\ absolute minimum is sometimes called a **global maximum\ global minimum**. The maximum and minimum values of f are called **the extreme values** of f.

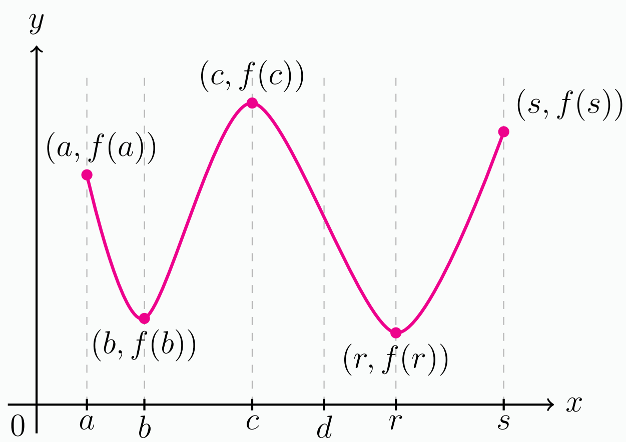
Local Extrema

Let c be in the domain of f.

- $f(c)$ is a **local maximum value** of f if
$$f(c) \geq f(x)$$
when x is near c.
- $f(c)$ is a **local minimum value** of f if
$$f(c) \leq f(x)$$
when x is near c.

13.2.1 Absolute and Local Extrema from a Graph

Example 13.1.



For the graph shown above, identify any absolute maximum values, absolute minimum values, local maximum values, and local minimum values.

Solution.
From the graph:

$$\boxed{\text{Absolute maximum value is } f(c).}$$

$$\boxed{\text{Absolute minimum value is } f(r).}$$

The local maximum values are

$$\boxed{f(a), f(c), \text{ and } f(s).}$$

The local minimum values are

$$\boxed{f(b) \text{ and } f(r).}$$

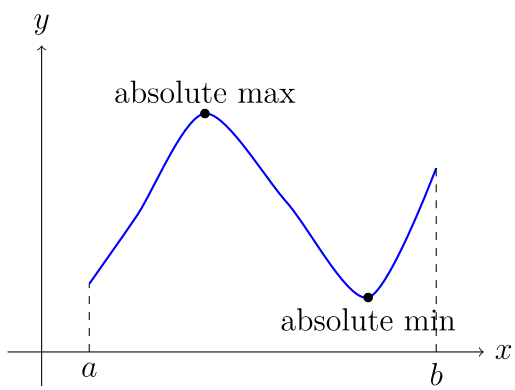
Warning

In the above example (13.1), $f(a)$ and $f(s)$ are local maximum values, assuming that we are allowed to consider the endpoints of the interval. If we can only consider interior points, then $f(c)$ is the only local maximum value.

13.2.2 The Extreme Value Theorem

The Extreme Value Theorem

If f is continuous on a closed interval $[a, b]$, then f attains an absolute maximum value $f(c)$ and an absolute minimum value $f(d)$ at some numbers c and d in $[a, b]$.



Warning

The Extreme Value Theorem requires two important hypotheses:

- The function must be **continuous**.
- The interval must be **closed and bounded**.

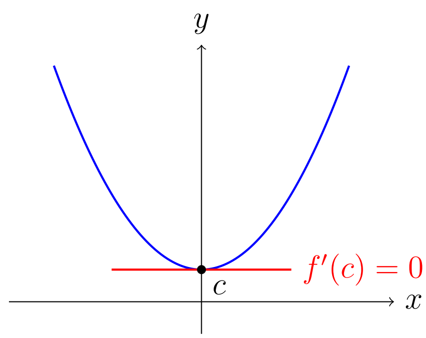
If either condition fails, the function may not have an absolute maximum or absolute minimum.

13.2.3 Fermat's Theorem

Fermat's Theorem

If f has a local maximum or local minimum at c, and if $f'(c)$ exists, then

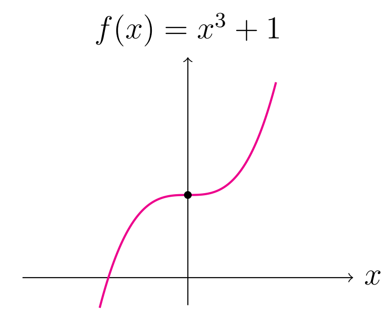
$$f'(c) = 0.$$



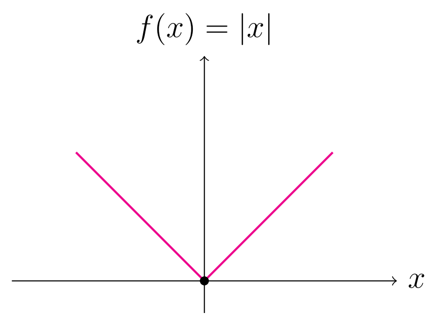
Warning

Be careful when using Fermat's Theorem.

- If $f'(c) = 0$, then c does **not necessarily** give a maximum or minimum.
- A function may have an extreme value even when $f'(c)$ does **not** exist.



$f'(0) = 0$, but no extreme value



minimum at $x =$, but $f'(0)$ does not exist

Critical Number

A number c in the domain of f is called a **critical number** of f if either

$$f'(c) = 0$$

or

$$f'(c) \text{ does not exist.}$$

Example 13.2.

Find the critical numbers of

$$f(x) = x^3 - 3x^2.$$

Solution.

By differentiating $f(x)$ with respect to x we get

$$f'(x) = 3x^2 - 6x = 3x(x - 2).$$

Then, by solving

$$f'(x) = 0 \qquad \text{for} \quad x$$

we get $x = 0, 2$. Therefore, the critical numbers are

$$\boxed{x = 0 \qquad \text{and} \qquad x = 2}$$

Example 13.3.

Find the critical numbers of

$$f(x) = x^{2/3}(x - 4).$$

Solution.

By differentiating using the Product Rule we get

$$\begin{aligned} f'(x) &= \frac{2}{3}x^{-1/3}(x - 4) + x^{2/3}(1) \\ &= x^{-1/3}\left[\frac{2}{3}(x - 4) + x\right] \\ &= \frac{5x - 8}{3x^{1/3}}. \end{aligned}$$

Critical numbers occur where
$f'(x) = 0$, or $f'(x)$ does not exist while $f(x)$ is defined.
Since

$$f'(x) = \frac{5x - 8}{3x^{1/3}},$$

we have

$$5x - 8 = 0$$

which gives

$$x = \frac{8}{5}.$$

Also, $f'(x)$ is undefined at

$$x = 0,$$

but

$$f(0) = 0$$

Thus, $x = 0$ is also a critical number.
Therefore, the critical numbers are

$$\boxed{x = 0 \qquad \text{and} \qquad x = \frac{8}{5}.}$$

13.2.4 The Closed Interval Method

The Closed Interval Method

To find the absolute maximum and absolute minimum values of a continuous function f on a closed interval $[a, b]$:

1. Find the values of f at the critical numbers of f in (a, b).
2. Find the values of f at the endpoints of the interval.
3. The largest of these values is the absolute maximum value.
4. The smallest of these values is the absolute minimum value.

Example 13.4.
Find the absolute maximum and absolute minimum values of

$$f(x) = x^3 - 3x^2 + 1$$

on the interval

$$[-1, 3].$$

Solution.
To find the absolute maximum and minimum values on $[-1, 3]$, we check the endpoints and the critical numbers. By differentiating $f(x)$ with respect to x we get

$$f'(x) = 3x^2 - 6x = 3x(x-2).$$

By solving

$$f'(x) = 0 \quad \text{for} \quad x$$

we get the critical numbers:

$$x = 0, \qquad x = 2.$$

Now evaluate f at the endpoints and critical numbers:

$$f(-1) = (-1)^3 - 3(-1)^2 + 1 = -3.$$

$$f(0) = 1.$$

$$f(2) = 2^3 - 3(2)^2 + 1 = 8 - 12 + 1 = -3.$$

$$f(3) = 3^3 - 3(3)^2 + 1 = 27 - 27 + 1 = 1.$$

Thus,

x	$f(x)$
-1	-3
0	1
2	-3
3	1

Therefore, the absolute maximum value is

$$\boxed{1}$$

and it occurs at

$$x = 0 \quad \text{and} \quad x = 3.$$

The absolute minimum value is

$$\boxed{-3}$$

and it occurs at

$$x = -1 \quad \text{and} \quad x = 2.$$

Example 13.5.
Find the absolute maximum and absolute minimum values of

$$f(x) = 12 + 4x - x^2$$

on the interval

$$[0, 5].$$

Solution.
By differentiating we get

$$f'(x) = 4 - 2x.$$

By solving

$$f'(x) = 0 \quad \text{for} \quad x$$

we get the critical number

$$x = 2.$$

Now evaluate $f(x)$ at the endpoints and the critical number:

$$f(0) = 12 + 4(0) - 0^2 = 12.$$

$$f(2) = 12 + 4(2) - 2^2 = 12 + 8 - 4 = 16.$$

$$f(5) = 12 + 4(5) - 5^2 = 12 + 20 - 25 = 7.$$

Thus,

x	$f(x)$
0	12
2	16
5	7

Therefore, the absolute maximum value is

$$\boxed{16}$$

and it occurs at

$$x = 2.$$

The absolute minimum value is

$$\boxed{7}$$

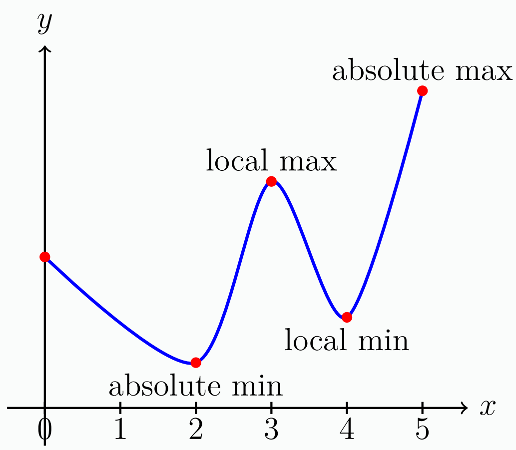
and it occurs at

$$x = 5.$$

Example 13.6.
Sketch the graph of a function on $[1, 5]$ with the following properties:

- absolute maximum at 5,
- absolute minimum at 2,
- local maximum at 3,
- local minima at 2 and 4.

Solution.



Example 13.7.
Show that 5 is a critical number of

$$g(x) = 2 + (x - 5)^3,$$

but g does not have a local extreme value at 5.

Solution.
By differentiating we get

$$g'(x) = 3(x-5)^2.$$

Thus,

$$g'(5) = 3(5-5)^2 = 0.$$

and therefore, 5 is a critical number of g.
Notice that

$$g'(x) > 0 \quad \text{for all} \quad x \neq 5.$$

Therefore, g is increasing on $\mathbb{R} \setminus \{5\}$. In particular,

$$g'(x) > 0 \quad \text{for } x < 5 \qquad \text{and} \qquad g'(x) > 0 \quad \text{for } x > 5.$$

Since the derivative does not change sign at $x = 5$, the function does not have a local maximum or local minimum at $x = 5$. That is, g has a saddle point at $x = 5$.
Therefore,

$$\boxed{\text{5 is a critical number, but } g \text{ has no local extreme value at 5.}}$$

Exercises

Find the critical numbers.

1. $f(x) = x^2 - 4x + 1$
2. $f(x) = x^3 - 12x + 2$
3. $f(x) = x^4 - 8x^2$
4. $f(x) = \sqrt[3]{x}(x - 1)$
5. $f(x) = \sin x + \cos x$
6. $g(t) = t^4 + t^3 + t^2 + 1$
7. $g(x) = \sqrt[3]{4 - x^2}$
8. $F(x) = x^{4/5}(x - 4)^2$
9. $g(\theta) = 4\theta - \tan\theta$
10. $f(\theta) = 2\cos\theta + \sin^2\theta$

Find the absolute maximum and absolute minimum values on the given interval.

11. $f(x) = x^2 - 4x + 1, \quad [0, 5]$
12. $f(x) = x^3 - 3x, \quad [-2, 2]$
13. $f(x) = x^4 - 4x^2, \quad [-3, 3]$
14. $f(x) = \sin x, \quad [0, 2\pi]$
15. $f(x) = \sqrt{x}, \quad [0, 9]$
16. $f(x) = 12 + 4x - x^2, \quad [0, 5]$
17. $f(x) = 5 + 54x - 2x^3, \quad [0, 4]$
18. $f(x) = 2x^3 - 3x^2 - 12x + 1, \quad [-2, 3]$

Sketch a graph satisfying the conditions.

19. A function on $[1, 5]$ with an absolute maximum at 5, an absolute minimum at 2, a local maximum at 3, and local minima at 2 and 4.

20. A function on $[1, 5]$ with an absolute maximum at 4, an absolute minimum at 5, a local maximum at 2, and a local minimum at 3.

21. A function on $[-1, 2]$ that has an absolute maximum but no local maximum.

22. A function on $[-1, 2]$ that has a local maximum but no absolute maximum.

23. A function on $[-1, 2]$ that has an absolute maximum but no absolute minimum.

24. A discontinuous function on $[-1, 2]$ that has both an absolute maximum and an absolute minimum.

Sketch the graph of f by hand and use your sketch to find the absolute and local maximum and minimum values of f.

25. $f(x) = \frac{1}{2}(3x - 1), \quad x \leq 3$

26. $f(x) = 2 - \frac{1}{3}x, \quad x \geq -2$

27. $f(x) = |x|$

28. $f(x) = 1 - \sqrt{x}$

29.
$$f(x) = \begin{cases} x^2, & -1 \leq x \leq 0, \\ 2 - 3x, & 0 < x \leq 1. \end{cases}$$

30.
$$f(x) = \begin{cases} 2x + 1, & 0 \leq x < 1, \\ 4 - 2x, & 1 \leq x \leq 3. \end{cases}$$

Conceptual Problems.

31. Give an example where $f'(c) = 0$ but c is not an extreme value.

32. Give an example where f has a local minimum at c, but $f'(c)$ does not exist.

33. Explain why continuity is necessary in the Extreme Value Theorem.

34. Can a function have two absolute maximum values? Explain.

35. Show that 5 is a critical number of

$$g(x) = 2 + (x - 5)^3,$$

but g does not have a local extreme value at 5.

36. Prove that

$$f(x) = x^{101} + x^{51} + x + 1$$

has neither a local maximum nor a local minimum.

Chapter 14

Derivatives and the Shape of a Graph

14.1. How Derivatives Affect the Shape of a Graph

Brief Description

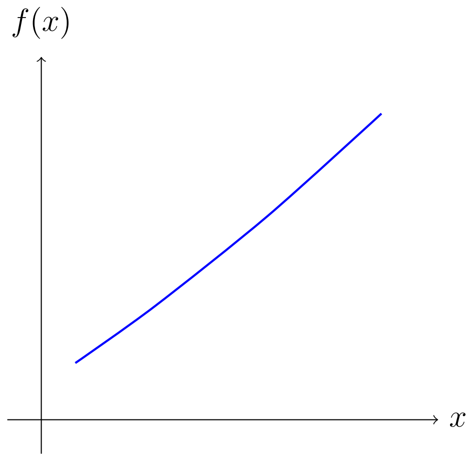
Many applications of calculus depend on our ability to understand the shape of a graph from information about its derivatives. Since $f'(x)$ represents the slope of the tangent line to the graph of $y = f(x)$, it tells us where the function is increasing or decreasing. The second derivative $f''(x)$ tells us how the slope is changing, which helps us determine concavity and inflection points.

Increasing and Decreasing Functions

Increasing/Decreasing Test

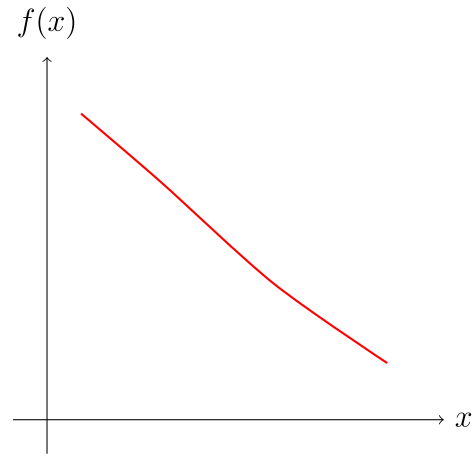
Let f be differentiable on an interval I.

1. If $f'(x) > 0$ for all x in I, then f is increasing on I.
2. If $f'(x) < 0$ for all x in I, then f is decreasing on I.



$f'(x) > 0$

Increasing



$f'(x) < 0$

Decreasing

Example 14.1.
Find the intervals on which

$$f(x) = x^3 - 3x^2 - 9x + 4$$

is increasing and decreasing.
Solution.
By differentiating we get

$$f'(x) = 3x^2 - 6x - 9 = 3(x-3)(x+1).$$

By solving

$$f'(x) = 0 \quad \text{for} \quad x$$

we get the critical numbers

$$x = -1 \qquad \text{and} \qquad x = 3.$$

Now test intervals:

$$(-\infty, -1), \qquad (-1, 3), \qquad (3, \infty).$$

Interval	$f'(x)$	Conclusion
$(-\infty, -1)$	$+$	increasing
$(-1, 3)$	$-$	decreasing
$(3, \infty)$	$+$	increasing

Therefore,

$$\boxed{\text{Increasing on } (-\infty, -1) \cup (3, \infty).}$$

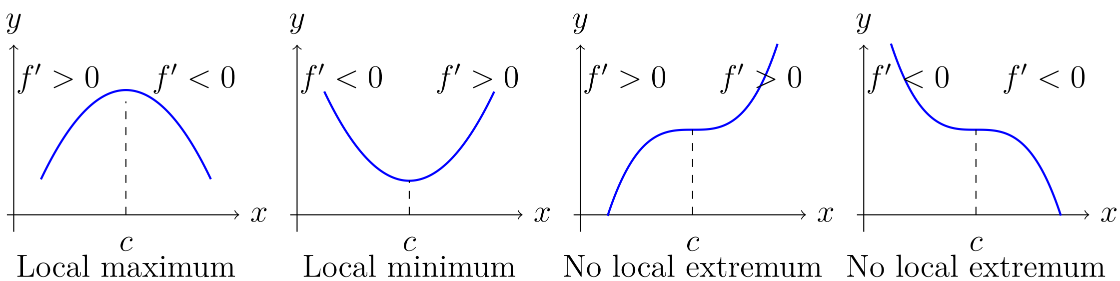
$$\boxed{\text{Decreasing on } (-1, 3).}$$

14.2. The First Derivative Test

The First Derivative Test

Suppose that c is a critical number of a continuous function f.

1. If f' changes from positive to negative at c, then f has a local maximum at c.
2. If f' changes from negative to positive at c, then f has a local minimum at c.
3. If f' does not change sign at c, then f has no local maximum or minimum at c.



Warning

If $f'(c) = 0$, then c is only a candidate for a local maximum or minimum. You must check whether f' changes sign.

Example 14.2.

Use the First Derivative Test to find the local maximum and local minimum values of

$$f(x) = x^3 - 3x^2 - 9x + 4.$$

Solution.
From the previous example, we found

$$f'(x) = 3(x-3)(x+1).$$

The critical numbers are

$$x = -1 \qquad \text{and} \qquad x = 3.$$

Now use the First Derivative Test.

Interval	$f'(x)$	Behavior of f
$(-\infty, -1)$	$+$	increasing
$(-1, 3)$	$-$	decreasing
$(3, \infty)$	$+$	increasing

At $x = -1$, $f'(x)$ changes from positive to negative, so f has a local maximum at $x = -1$.

$$f(-1) = (-1)^3 - 3(-1)^2 - 9(-1) + 4 = -1 - 3 + 9 + 4 = 9.$$

At $x = 3$, $f'(x)$ changes from negative to positive, so f has a local minimum at $x = 3$.

$$f(3) = 3^3 - 3(3)^2 - 9(3) + 4 = 27 - 27 - 27 + 4 = -23.$$

Therefore,

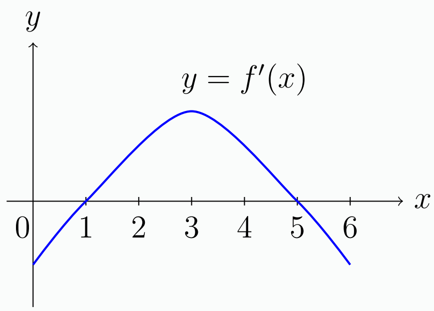
$$\boxed{\text{Local maximum value is } 9 \text{ at } x = -1.}$$

$$\boxed{\text{Local minimum value is } -23 \text{ at } x = 3.}$$

Using the Graph of f'

Example 14.3.

The graph of the derivative f' of a function f is shown.



1. On what intervals is f increasing or decreasing?
2. At what values of x does f have a local maximum or local minimum?

Solution.

Since the graph shown is the graph of $f'(x)$, we use the sign of $f'(x)$.
From the graph,

$$f'(x) < 0 \quad \text{on } (0, 1),$$

$$f'(x) > 0 \quad \text{on } (1, 5),$$

and

$$f'(x) < 0 \quad \text{on } (5, 6).$$

Therefore,

$$\boxed{f \text{ is increasing on } (1, 5).}$$

$$\boxed{f \text{ is decreasing on } (0, 1) \cup (5, 6).}$$

Next, local extrema occur where $f'(x)$ changes sign.
At $x = 1$, $f'(x)$ changes from negative to positive, so f has a local minimum.
At $x = 5$, $f'(x)$ changes from positive to negative, so f has a local maximum.
Thus,

$$\boxed{f \text{ has a local minimum at } x = 1.}$$

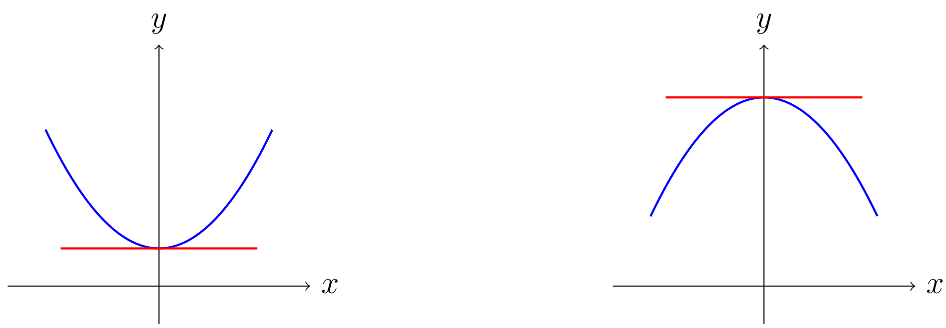
$$\boxed{f \text{ has a local maximum at } x = 5.}$$

14.3. Concavity

The first derivative tells us whether a function is increasing or decreasing. The second derivative tells us how the graph bends.

Concavity

If the graph of f lies above its tangent lines on an interval I, then f is **concave upward** on I.
If the graph of f lies below its tangent lines on an interval I, then f is **concave downward** on I.



Concave upward
$f''(x) > 0$

Concave downward
$f''(x) < 0$

14.3.1 The Concavity Test

Concavity Test

Let f be twice differentiable on an interval I.

1. If $f''(x) > 0$ for all x in I, then the graph of f is concave upward on I.
2. If $f''(x) < 0$ for all x in I, then the graph of f is concave downward on I.

Example 14.4.

Find the intervals of concavity of

$$f(x) = x^3 - 3x^2 - 9x + 4.$$

Solution.

By differentiating twice we get

$$f'(x) = 3x^2 - 6x - 9,$$

and

$$f''(x) = 6x - 6 = 6(x-1).$$

By solving

$$f''(x) = 0 \quad \text{for} \quad x$$

we get

$$x = 1.$$

Now test the intervals

$$(-\infty, 1) \qquad \text{and} \qquad (1, \infty).$$

Interval	$f''(x)$	Concavity
$(-\infty, 1)$	$-$	concave down
$(1, \infty)$	$+$	concave up

Therefore,

$$\boxed{f \text{ is concave down on } (-\infty, 1).}$$

$$\boxed{f \text{ is concave up on } (1, \infty).}$$

Since the concavity changes at $x = 1$, the function has an inflection point there. Its y-coordinate is

$$f(1) = 1 - 3 - 9 + 4 = -7.$$

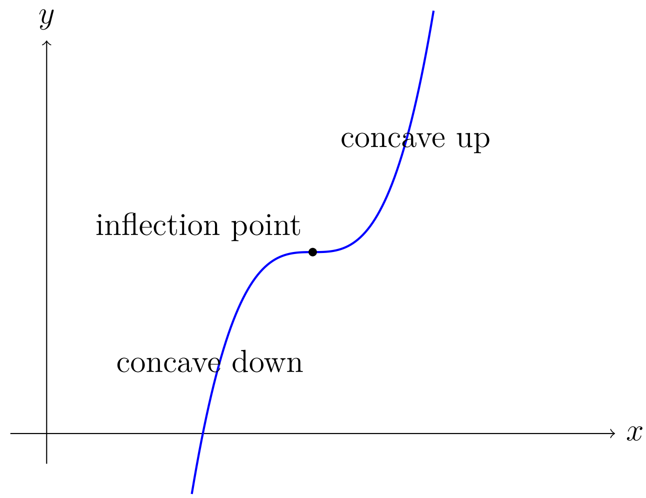
Thus, the inflection point is

$$\boxed{(1, -7).}$$

14.3.2 Inflection Points

Inflection Point

A point $P \equiv (c, f(c))$ on the curve $y = f(x)$ is called an **inflection point** if f is continuous there and the curve changes from concave upward to concave downward, or from concave downward to concave upward, at P.



Example 14.5.

Find the inflection point(s) of

$$f(x) = x^3 - 3x^2 - 9x + 4.$$

Solution.

By differentiating twice we get

$$f'(x) = 3x^2 - 6x - 9,$$

$$f''(x) = 6x - 6 = 6(x - 1).$$

By solving

$$f''(x) = 0 \quad \text{for} \quad x$$

we get

$$x = 1.$$

So, the only possible inflection point occurs at $x = 1$.

Because

$$f''(x) < 0 \quad \text{for } x < 1,$$

and

$$f''(x) > 0 \quad \text{for } x > 1,$$

the concavity changes from down to up at $x = 1$. Therefore, $x = 1$ is an inflection point. To find the corresponding y-coordinate:

$$f(1) = 1^3 - 3(1)^2 - 9(1) + 4 = 1 - 3 - 9 + 4 = -7.$$

Hence, the inflection point is

$$\boxed{(1, -7).}$$

Warning

A point where

$$f''(c) = 0$$

is not automatically an inflection point. The concavity must actually change at c.

Example 14.6.

Find the inflection point(s) of

$$g(x) = x^4.$$

Solution.

by differentiating twice we get

$$g'(x) = 4x^3,$$

$$g''(x) = 12x^2.$$

Thus, the only possible inflection point is

$$x = 0.$$

Notice that

$$g''(x) > 0$$

for all $x \neq 0$.
Thus, the graph is concave up on both sides of $x = 0$, so the concavity does *not* change at $x = 0$.
Therefore, $x = 0$ is *not* an inflection point.
Hence,

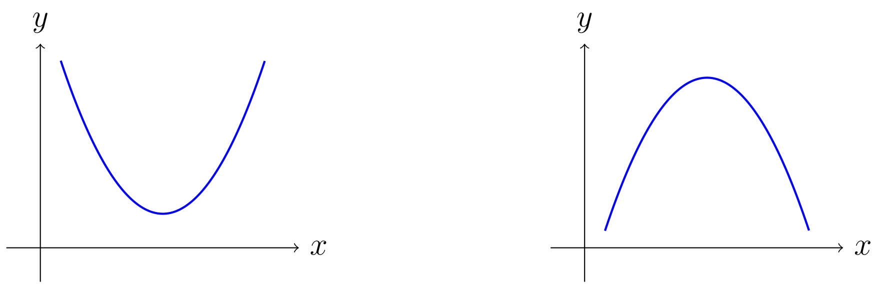
$$\boxed{\text{There are no inflection points.}}$$

14.4. The Second Derivative Test

The Second Derivative Test

Suppose f'' is continuous near c and $f'(c) = 0$.

1. If $f''(c) > 0$, then f has a local minimum at c.
2. If $f''(c) < 0$, then f has a local maximum at c.
3. If $f''(c) = 0$, then the test is inconclusive.



$f''(c) > 0$
local minimum

$f''(c) < 0$
local maximum

Example 14.7.
Use the Second Derivative Test to classify the critical points of

$$f(x) = x^4 - 2x^2 + 3.$$

Solution.
By differentiating we get

$$f'(x) = 4x^3 - 4x = 4x(x - 1)(x + 1).$$

Thus, the critical numbers are

$$x = -1, \qquad x = 0, \qquad x = 1.$$

Next, compute the second derivative:

$$f''(x) = 12x^2 - 4.$$

Evaluate f'' at each critical number:

$$f''(-1) = 12(1) - 4 = 8 > 0,$$

so f has a local minimum at $x = -1$, and

$$f(-1) = (-1)^4 - 2(-1)^2 + 3 = 1 - 2 + 3 = 2.$$

Next,

$$f''(0) = 12(0) - 4 = -4 < 0,$$

so f has a local maximum at $x = 0$, and

$$f(0) = 3.$$

Finally,

$$f''(1) = 12(1) - 4 = 8 > 0,$$

so f has a local minimum at $x = 1$, and

$$f(1) = 1 - 2 + 3 = 2.$$

Therefore,

$$\boxed{\text{Local minimum values: 2 at } x = -1 \text{ and } x = 1.}$$

$$\boxed{\text{Local maximum value: 3 at } x = 0.}$$

Curve Sketching Using Derivatives

Summary for Curve Sketching

To sketch the graph of a function using derivatives:

1. Find the domain.
2. Find critical numbers by solving $f'(x) = 0$ or finding where $f'(x)$ does not exist.
3. Use f' to determine intervals of increase and decrease.
4. Use the First Derivative Test to classify local extrema.
5. Find where $f''(x) = 0$ or where $f''(x)$ does not exist.
6. Use f'' to determine concavity.
7. Find inflection points where concavity changes.

Example 14.8.

For

$$f(x) = \frac{1}{2}x^4 - 4x^2 + 3,$$

find:

(a) the intervals on which f is increasing or decreasing,

(b) the local maximum and local minimum values,

(c) the intervals of concavity,

(d) the inflection points,

(e) sketch the graph.

Solution.

(a) By differentiating we get

$$f'(x) = 2x^3 - 8x = 2x(x^2 - 4) = 2x(x-2)(x+2).$$

Thus, the critical numbers are

$$x = -2, \qquad x = 0, \qquad x = 2.$$

The sign of $f'(x)$ is summarized below:

Interval	$f'(x)$	Behavior of f
$(-\infty, -2)$	$-$	decreasing
$(-2, 0)$	$+$	increasing
$(0, 2)$	$-$	decreasing
$(2, \infty)$	$+$	increasing

Therefore,

$$\boxed{f \text{ is increasing on } (-2, 0) \cup (2, \infty).}$$

$$\boxed{f \text{ is decreasing on } (-\infty, -2) \cup (0, 2).}$$

(b) At $x = -2$, $f'(x)$ changes from negative to positive, so f has a local minimum.

At $x = 0$, $f'(x)$ changes from positive to negative, so f has a local maximum.

At $x = 2$, $f'(x)$ changes from negative to positive, so f has another local minimum.

Evaluating the function at these values:

$$f(-2) = \frac{1}{2}(-2)^4 - 4(-2)^2 + 3 = 8 - 16 + 3 = -5,$$

$$f(0) = 3,$$

and

$$f(2) = \frac{1}{2}(2)^4 - 4(2)^2 + 3 = 8 - 16 + 3 = -5.$$

Thus,

$$\boxed{\text{The local maximum value is } 3 \text{ at } x = 0.}$$

$$\boxed{\text{The local minimum value is } -5 \text{ at } x = -2 \text{ and } x = 2.}$$

The corresponding local extrema are

$$\boxed{(0, 3), \qquad (-2, -5), \qquad (2, -5).}$$

(c) By differentiating again we get

$$f''(x) = 6x^2 - 8.$$

By solving $f''(x) = 0$ for x we get

$$x = \pm\frac{2}{\sqrt{3}} = \pm\frac{2\sqrt{3}}{3}.$$

The sign of $f''(x)$ is summarized below:

Interval	$f''(x)$	Concavity
$\left(-\infty, -\frac{2\sqrt{3}}{3}\right)$	$+$	concave up
$\left(-\frac{2\sqrt{3}}{3}, \frac{2\sqrt{3}}{3}\right)$	$-$	concave down
$\left(\frac{2\sqrt{3}}{3}, \infty\right)$	$+$	concave up

Therefore,

$$\boxed{f \text{ is concave up on } \left(-\infty, -\frac{2\sqrt{3}}{3}\right) \cup \left(\frac{2\sqrt{3}}{3}, \infty\right).}$$

$$\boxed{f \text{ is concave down on } \left(-\frac{2\sqrt{3}}{3}, \frac{2\sqrt{3}}{3}\right).}$$

(d) The concavity changes at

$$x = \pm\frac{2\sqrt{3}}{3}.$$

Since

$$x^2 = \frac{4}{3} \qquad \text{and} \qquad x^4 = \frac{16}{9},$$

we have

$$\begin{aligned} f\left(\pm\frac{2\sqrt{3}}{3}\right) &= \frac{1}{2}\left(\frac{16}{9}\right) - 4\left(\frac{4}{3}\right) + 3 \\ &= \frac{8}{9} - \frac{16}{3} + 3 \\ &= -\frac{13}{9}. \end{aligned}$$

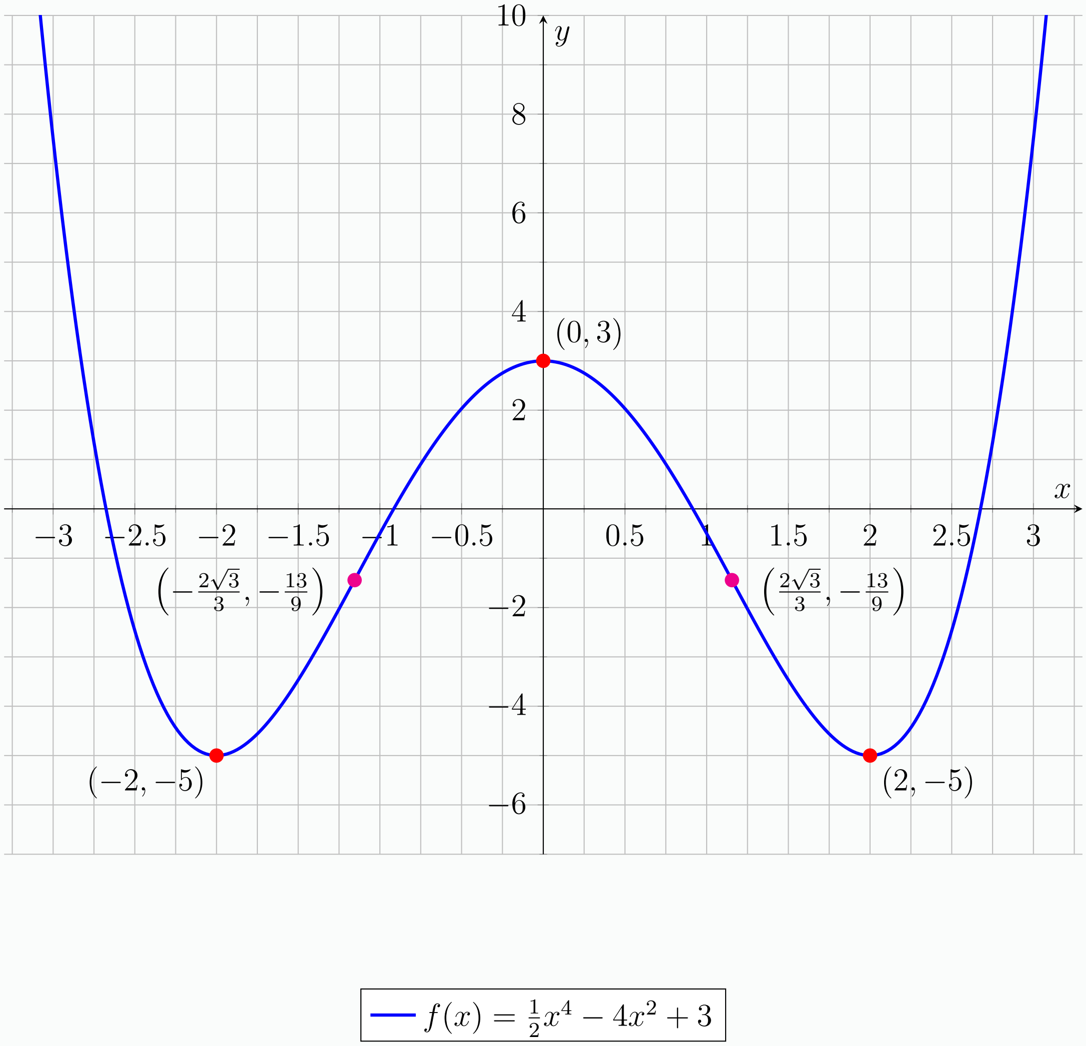
Therefore, the inflection points are

$$\boxed{\left(-\frac{2\sqrt{3}}{3}, -\frac{13}{9}\right) \quad \text{and} \quad \left(\frac{2\sqrt{3}}{3}, -\frac{13}{9}\right).}$$

(e) The function is even because

$$f(-x) = f(x),$$

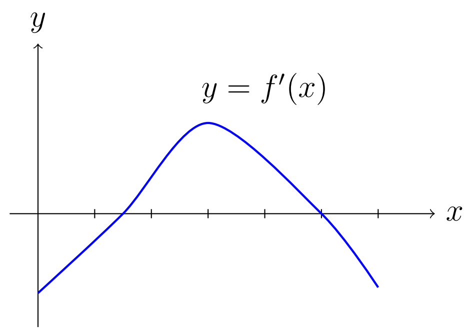
so its graph is symmetric about the y-axis.



Exercises

Derivative Graph Interpretation

1. The graph of f' is shown below.



   (a) On what intervals is f increasing or decreasing?

   (b) At what values of x does f have a local maximum or local minimum?

Increasing/Decreasing, Local Extrema, Concavity, and Inflection Points

For each function, find:

(a) the intervals on which f is increasing or decreasing,

(b) the local maximum and local minimum values,

(c) the intervals of concavity and the inflection points.

2. $f(x) = x^3 - 3x^2 - 9x + 4$
3. $f(x) = 2x^3 - 9x^2 + 12x - 3$
4. $f(x) = x^4 - 2x^2 + 3$
5. $f(x) = \dfrac{x}{x^2 + 1}$
6. $f(x) = \sin x + \cos x, \quad 0 \le x \le 2\pi$
7. $f(x) = x^3 - 12x + 2$
8. $f(x) = 36x + 3x^2 - 2x^3$
9. $f(x) = \dfrac{1}{2}x^4 - 4x^2 + 3$

Second Derivative Test

9. Find the critical numbers of
$$f(x) = x^4(x-1)^3.$$
What does the Second Derivative Test tell you about the behavior of f at these critical numbers? What does the First Derivative Test tell you?

10. Suppose f'' is continuous on $(-\infty, \infty)$.
    (a) If $f'(2) = 0$ and $f''(2) = -5$, what can you say about f?
    (b) If $f'(6) = 0$ and $f''(6) = 0$, what can you say about f?

Curve Sketching

11. Sketch the graph of
$$f(x) = x^3 - 12x + 2.$$

12. Sketch the graph of
$$f(x) = 36x + 3x^2 - 2x^3.$$

13. Sketch the graph of
$$f(x) = \frac{1}{2}x^4 - 4x^2 + 3.$$

Conceptual Problems

14. Suppose
$$f(3) = 2, \qquad f'(3) = \frac{1}{2},$$
and
$$f'(x) > 0, \qquad f''(x) < 0$$
for all x.
    (a) Sketch a possible graph for f.
    (b) How many solutions does the equation $f(x) = 0$ have? Explain.
    (c) Is it possible that $f'(2) = \frac{1}{3}$? Explain.

15. Suppose
$$f'(x) = (x+1)^2(x-3)^5(x-6)^4.$$
On what intervals is f increasing?

16. Show that

$$\tan x > x$$

for

$$0 < x < \frac{\pi}{2}.$$

17. Prove that, for all $x > 1$,

$$2\sqrt{x} > 3 - \frac{1}{x}.$$

Chapter 15

Limits at Infinity: Horizontal Asymptotes

15.1. Limits at Infinity; Horizontal Asymptotes

Brief Description

In previous sections, we studied limits as x approaches a finite number. In this section we investigate the behavior of functions as x becomes arbitrarily large or arbitrarily small.

These limits help us describe the *end behavior* of functions and are used to determine horizontal asymptotes. We will study limits at infinity, infinite limits at infinity, horizontal asymptotes, and the end behavior of polynomial and rational functions.

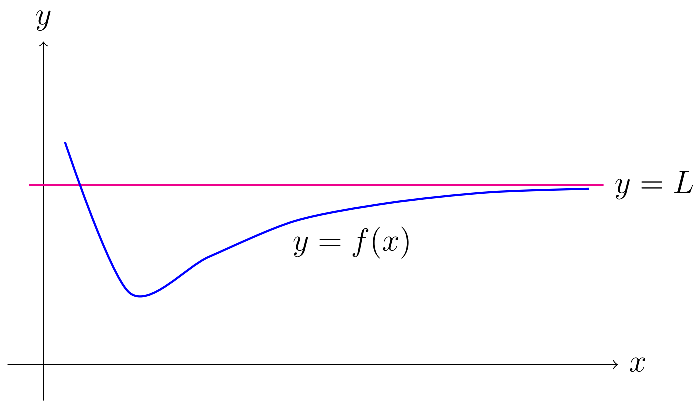
15.1.1 Limits at Infinity

Intuitive Definition of a Limit at Infinity

Let f be a function defined on some interval (a, ∞).
We write

$$\lim_{x\to\infty} f(x) = L$$

if the values of $f(x)$ can be made arbitrarily close to L by taking x sufficiently large.

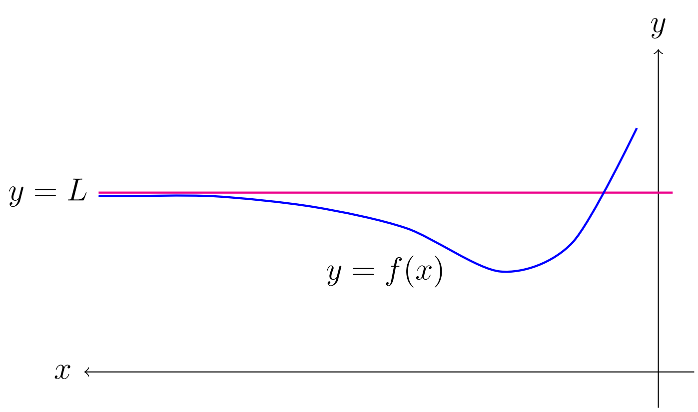


Limit at Negative Infinity

Let f be defined on an interval $(-\infty, a)$.
We write

$$\lim_{x\to-\infty} f(x) = L$$

if the values of $f(x)$ can be made arbitrarily close to L by taking x sufficiently negative.



15.1.2 Horizontal Asymptotes

Horizontal Asymptote

The line

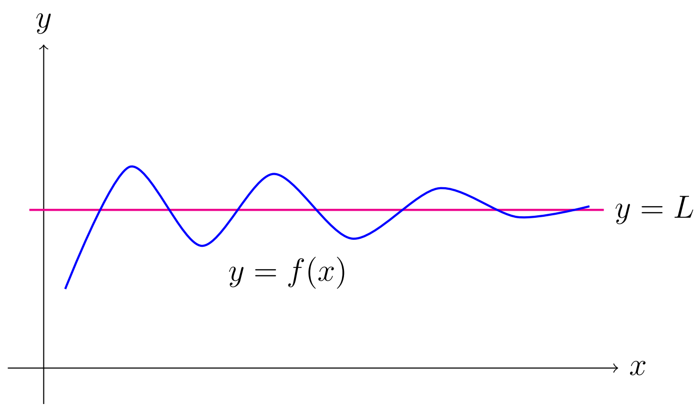
$$y = L$$

is called a **horizontal asymptote** of the graph of f if

$$\lim_{x\to\infty} f(x) = L \qquad \text{or} \qquad \lim_{x\to-\infty} f(x) = L.$$

Warning

A graph may cross a horizontal asymptote many times.
A horizontal asymptote only describes the behavior of a function as $x \to \pm\infty$.



15.1.3 Basic Limits at Infinity

A Fundamental Limit

If $r > 0$, then

$$\lim_{x\to\infty} \frac{1}{x^r} = 0.$$

Similarly,

$$\lim_{x\to-\infty} \frac{1}{x^r} = 0$$

whenever the expression is defined.

Example 15.1.

Determine

$$\lim_{x\to\infty} \frac{1}{x^3}.$$

Solution.

$$\lim_{x\to\infty} \frac{1}{x^3} = \frac{1}{\infty^3} = 0.$$

Thus,

$$\boxed{\lim_{x\to\infty} \frac{1}{x^3} = 0}.$$

15.1.4 Infinite Limits at Infinity

Infinite Limits at Infinity

The notation

$$\lim_{x\to\infty} f(x) = \infty$$

means that the values of $f(x)$ become arbitrarily large as x becomes arbitrarily large. Likewise,

$$\lim_{x\to\infty} f(x) = -\infty,$$

$$\lim_{x\to-\infty} f(x) = \infty,$$

and

$$\lim_{x\to-\infty} f(x) = -\infty$$

have analogous meanings.

Example 15.2.

Evaluate

$$\lim_{x\to\infty} (x^2 - 4x).$$

Solution.

$$\lim_{x\to\infty} (x^2 - 4x) = \lim_{x\to\infty} x^2 = (\infty)^2 = \infty.$$

$$\boxed{\lim_{x\to\infty} (x^2 - 4x) = \infty.}$$

Example 15.3.

Evaluate

$$\lim_{x\to-\infty} (x^2 - 4x).$$

Solution.

$$\lim_{x\to\infty} (x^2 - 4x) = \lim_{x\to\infty} x^2 = (-\infty)^2 = \infty.$$

$$\boxed{\lim_{x\to-\infty} (x^2 - 4x) = \infty.}$$

15.1.5 Limit Laws at Infinity

Limit Laws at Infinity

Suppose

$$\lim_{x\to\infty} f(x) = L \qquad \text{and} \qquad \lim_{x\to\infty} g(x) = M.$$

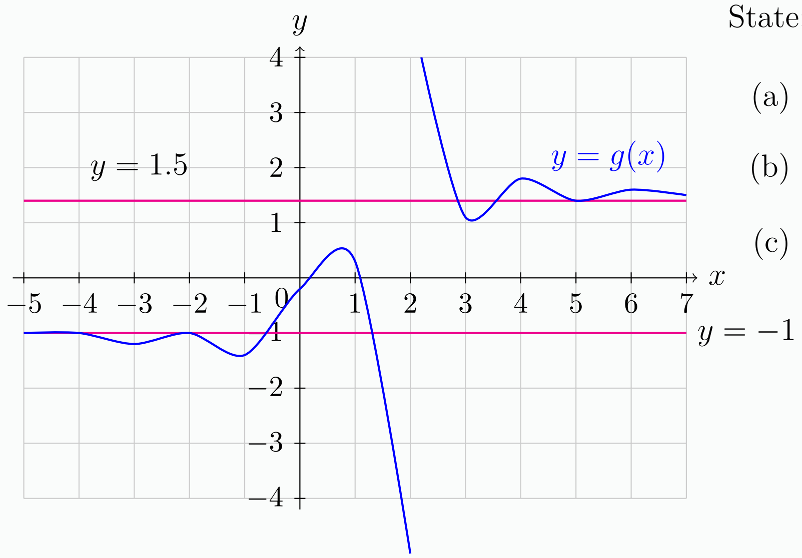
Then the usual limit laws remain valid:

$$\begin{aligned}
\lim_{x\to\infty}[f(x)+g(x)] &= L+M,\\
\lim_{x\to\infty}[f(x)-g(x)] &= L-M,\\
\lim_{x\to\infty}[f(x)g(x)] &= LM,\\
\lim_{x\to\infty}\frac{f(x)}{g(x)} &= \frac{L}{M}, \qquad \text{given } M\neq 0.
\end{aligned}$$

Example: Reading Limits from a Graph

Example 15.4.

The graph of g is shown below.



State:

(a) $\lim_{x\to\infty} g(x)$

(b) $\lim_{x\to-\infty} g(x)$

(c) The horizontal asymptote(s)

Solution.

(a) $\lim_{x\to\infty} g(x) = 1.5$

(b) $\lim_{x\to-\infty} g(x) = -1$

(c) The horizontal asymptotes are

$$y=-1 \quad \text{and} \quad y=1.5.$$

15.1.6 Limits of Rational Functions at Infinity

Rational Functions at Infinity

Suppose

$$R(x) = \frac{a_n x^n + \cdots + a_0}{b_m x^m + \cdots + b_0}.$$

Then:

1. If $n < m$, then
$$\lim_{x \to \pm\infty} R(x) = 0.$$
2. If $n = m$, then
$$\lim_{x \to \pm\infty} R(x) = \frac{a_n}{b_m}.$$
3. If $n > m$, then
$$\lim_{x \to \pm\infty} R(x)$$
does not approach a finite number and may be $\pm\infty$ or fail to exist.

Example 15.5.

Evaluate

$$\lim_{x \to \infty} \frac{3x - 2}{2x + 1}.$$

Solution.

$$\begin{aligned}
\lim_{x \to \infty} \frac{3x - 2}{2x + 1} &= \lim_{x \to \infty} \frac{3x}{2x} \\
&= \lim_{x \to \infty} \frac{3}{2} \\
&= \frac{3}{2}.
\end{aligned}$$

$$\boxed{\lim_{x \to \infty} \frac{3x - 2}{2x + 1} = \frac{3}{2}.}$$

Example 15.6.
Evaluate

$$\lim_{x\to\infty}\frac{1-x^2}{x^3-x+1}.$$

Solution.

$$\begin{aligned}\lim_{x\to\infty}\frac{1-x^2}{x^3-x+1} &= \lim_{x\to\infty}\frac{-x^2}{x^3}\\ &= \lim_{x\to\infty}-\frac{1}{x}\\ &= 0.\end{aligned}$$

$$\boxed{\lim_{x\to\infty}\frac{1-x^2}{x^3-x+1}=0.}$$

Example 15.7.
Evaluate

$$\lim_{x\to\infty}\frac{1+x^6}{x^4+1}.$$

Solution.

$$\begin{aligned}\lim_{x\to\infty}\frac{1+x^6}{x^4+1} &= \lim_{x\to\infty}\frac{x^6}{x^4}\\ &= \lim_{x\to\infty}x^2\\ &= \infty.\end{aligned}$$

$$\boxed{\lim_{x\to\infty}\frac{1+x^6}{x^4+1}=\infty.}$$

15.1.7 Limits Involving Radicals

Example 15.8.
Evaluate

$$\lim_{x\to\infty}\frac{\sqrt{x+3x^2}}{4x-1}.$$

Solution.

$$\begin{aligned}
\lim_{x\to\infty} \frac{\sqrt{x+3x^2}}{4x-1} &= \lim_{x\to\infty} \frac{\sqrt{3x^2}}{4x} \\
&= \lim_{x\to\infty} \frac{\sqrt{3}x}{4x} \\
&= \lim_{x\to\infty} \frac{\sqrt{3}}{4} \\
&= \frac{\sqrt{3}}{4}.
\end{aligned}$$

$$\boxed{\lim_{x\to\infty} \frac{\sqrt{x+3x^2}}{4x-1} = \frac{\sqrt{3}}{4}.}$$

Example 15.9.
Evaluate

$$\lim_{x\to-\infty} \frac{\sqrt{1+4x^6}}{2-x^3}.$$

Solution.

$$\begin{aligned}
\lim_{x\to-\infty} \frac{\sqrt{1+4x^6}}{2-x^3} &= \lim_{x\to-\infty} \frac{\sqrt{4x^6}}{-x^3} \\
&= \lim_{x\to-\infty} \frac{2x^3}{-x^3} \\
&= \lim_{x\to-\infty} (-2) \\
&= -2.
\end{aligned}$$

$$\boxed{\lim_{x\to-\infty} \frac{\sqrt{1+4x^6}}{2-x^3} = -2.}$$

Example 15.10.
Evaluate

$$\lim_{x\to\infty} \left(\sqrt{9x^2+x} - 3x\right).$$

Solution.

$$
\begin{aligned}
\lim_{x\to\infty}\left(\sqrt{9x^2+x}-3x\right) &= \lim_{x\to\infty}\left[\left(\sqrt{9x^2+x}-3x\right)\cdot\frac{\left(\sqrt{9x^2+x}+3x\right)}{\left(\sqrt{9x^2+x}+3x\right)}\right] \\
&= \lim_{x\to\infty}\frac{\left(\sqrt{9x^2+x}\right)^2-(3x)^2}{\sqrt{9x^2+x}+3x} \\
&= \lim_{x\to\infty}\frac{9x^2+x-9x}{\sqrt{9x^2+x}+3x} \\
&= \lim_{x\to\infty}\frac{x}{\sqrt{9x^2+x}+3x} \\
&= \lim_{x\to\infty}\frac{x}{\sqrt{9x^2}+3x} \\
&= \lim_{x\to\infty}\frac{x}{3x+3x} \\
&= \lim_{x\to\infty}\frac{x}{6x} \\
&= \lim_{x\to\infty}\frac{1}{6} \\
&= \frac{1}{6}.
\end{aligned}
$$

$$
\boxed{\lim_{x\to\infty}\left(\sqrt{9x^2+x}-3x\right)=\frac{1}{6}.}
$$

Example 15.11.

Evaluate

$$
\lim_{x\to\infty}\left(\sqrt{x^2+ax}-\sqrt{x^2+bx}\right).
$$

Solution.

$$
\begin{aligned}
\lim_{x\to\infty}\left(\sqrt{x^2+ax}-\sqrt{x^2+bx}\right) &= \lim_{x\to\infty}\left[\left(\sqrt{x^2+ax}-\sqrt{x^2+bx}\right)\cdot\frac{\left(\sqrt{x^2+ax}+\sqrt{x^2+bx}\right)}{\left(\sqrt{x^2+ax}+\sqrt{x^2+bx}\right)}\right] \\
&= \lim_{x\to\infty}\frac{x^2+ax-(x^2+bx)}{\sqrt{x^2+ax}+\sqrt{x^2+bx}} \\
&= \lim_{x\to\infty}\frac{ax-bx}{\sqrt{x^2+ax}+\sqrt{x^2+bx}} \\
&= \lim_{x\to\infty}\frac{ax-bx}{\sqrt{x^2}+\sqrt{x^2}}
\end{aligned}
$$

$$
\begin{aligned}
&= \lim_{x\to\infty} \frac{ax - bx}{x + x} \\
&= \lim_{x\to\infty} \frac{ax - bx}{2x} \\
&= \lim_{x\to\infty} \frac{a - b}{2} \\
&= \frac{a - b}{2}.
\end{aligned}
$$

$$
\boxed{\lim_{x\to\infty} \left(\sqrt{x^2 + ax} - \sqrt{x^2 + bx}\right) = \frac{a - b}{2}.}
$$

A Useful Algebraic Technique

Warning

Whenever radicals are added or subtracted, consider multiplying by a conjugate. This often converts an indeterminate form into a simpler limit.

15.1.8 Oscillating Functions

Example 15.12.

Determine whether

$$\lim_{x\to\infty} \cos x$$

exists.

Solution.

As $x \to \infty$, the function $\cos x$ continues to oscillate between

$$-1 \qquad \text{and} \qquad 1.$$

For example, along the sequence

$$x_n = 2\pi n,$$

we have

$$\cos(x_n) = 1.$$

But along the sequence

$$y_n = (2n + 1)\pi,$$

we have

$$\cos(y_n) = -1.$$

Since $\cos x$ approaches different values along different sequences as $x \to \infty$, it does not approach a single number.
Therefore,

$$\boxed{\lim_{x\to\infty} \cos x \text{ does not exist.}}$$

Warning

A function may be bounded and still fail to have a limit at infinity.
Oscillation prevents the existence of a limit.

Example 15.13.
Evaluate

$$\lim_{x\to\infty} x \sin\left(\frac{1}{x}\right).$$

Solution.
Let

$$u = \frac{1}{x}.$$

As $x \to \infty$ we have

$$u \to 0.$$

Then

$$x \sin\left(\frac{1}{x}\right) = \frac{\sin u}{u}.$$

Thus,

$$\lim_{x\to\infty} x \sin\left(\frac{1}{x}\right) = \lim_{u\to 0} \frac{\sin u}{u}.$$

We know that

$$\lim_{u\to 0} \frac{\sin u}{u} = 1.$$

Therefore,

$$\boxed{\lim_{x\to\infty} x \sin\left(\frac{1}{x}\right) = 1.}$$

Example 15.14.

Evaluate

$$\lim_{x\to\infty} \sqrt{x}\sin\left(\frac{1}{x}\right).$$

Solution.

Let

$$u = \frac{1}{x}.$$

Then as $x \to \infty$ we have

$$u \to 0,$$

and

$$\sqrt{x}\sin\left(\frac{1}{x}\right) = \frac{\sin u}{1/\sqrt{u}} = \sqrt{u}\cdot\frac{\sin u}{u}.$$

Thus,

$$\lim_{x\to\infty} \sqrt{x}\sin\left(\frac{1}{x}\right) = \left(\lim_{u\to 0}\sqrt{u}\right)\left(\lim_{u\to 0}\frac{\sin u}{u}\right).$$

Using the standard limit

$$\lim_{u\to 0}\frac{\sin u}{u} = 1,$$

we obtain

$$0 \cdot 1 = 0.$$

Therefore,

$$\boxed{\lim_{x\to\infty} \sqrt{x}\sin\left(\frac{1}{x}\right) = 0.}$$

15.1.9 Determining Horizontal Asymptotes

Finding Horizontal Asymptotes

To determine horizontal asymptotes:

1. Compute
$$\lim_{x\to\infty} f(x).$$
2. Compute
$$\lim_{x\to-\infty} f(x).$$
3. If either limit equals L, then
$$y = L$$
is a horizontal asymptote.

Example 15.15.

Find all horizontal asymptotes of
$$y = \frac{5+4x}{x+3}.$$

Solution.

We see that
$$\lim_{x\to-\infty} y = \lim_{x\to-\infty}\left(\frac{5+4x}{x+3}\right) = \lim_{x\to-\infty}\left(\frac{4x}{x}\right) = \lim_{x\to-\infty} 4 = 4$$
and
$$\lim_{x\to\infty} y = \lim_{x\to\infty}\left(\frac{5+4x}{x+3}\right) = \lim_{x\to\infty}\left(\frac{4x}{x}\right) = \lim_{x\to\infty} 4 = 4.$$
Thus,
$$y = 4$$
is the only horizontal asymptote.
$$\boxed{y = 4}$$

Example 15.16.

Find all horizontal asymptotes of

$$y = \frac{2x^2+1}{3x^2+2x-1}.$$

Solution. We see that

$$\lim_{x\to-\infty} y = \lim_{x\to-\infty} \frac{2x^2+1}{3x^2+2x-1} = \lim_{x\to-\infty} \frac{2x^2}{3x^2} = \lim_{x\to-\infty} \frac{2}{3} = \frac{2}{3}$$

and

$$\lim_{x\to\infty} y = \lim_{x\to\infty} \frac{2x^2+1}{3x^2+2x-1} = \lim_{x\to\infty} \frac{2x^2}{3x^2} = \lim_{x\to\infty} \frac{2}{3} = \frac{2}{3}.$$

Therefore, the function has one horizontal asymptote:

$$\boxed{y = \frac{2}{3}.}$$

15.1.10 End Behavior of Polynomial Functions

End Behavior of Polynomials

For large values of $|x|$, a polynomial behaves like its leading term.
That is,

$$a_n x^n + \cdots + a_0$$

has the same end behavior as

$$a_n x^n.$$

Example 15.17.

Determine the end behavior of

$$f(x) = 2x^3 - x^4.$$

Solution.

The highest-degree term of

$$f(x) = 2x^3 - x^4$$

is

$$-x^4.$$

Since the leading term has even degree and a negative coefficient, it determines the end

behavior.
As $x \to \infty$,

$$2x^3 - x^4 \longrightarrow -\infty,$$

because the $-x^4$ term dominates.
Similarly, as $x \to -\infty$,

$$2x^3 - x^4 \longrightarrow -\infty,$$

since

$$-x^4 \to -\infty$$

and it grows much faster than the cubic term.
Therefore,

$$\boxed{\lim_{x\to\infty}(2x^3 - x^4) = -\infty}$$

and

$$\boxed{\lim_{x\to-\infty}(2x^3 - x^4) = -\infty.}$$

Thus, both ends of the graph fall downward.

Example 15.18.
Determine the end behavior of

$$f(x) = x^3(x+2)^2(x-1).$$

Solution.
The function is

$$f(x) = x^3(x+2)^2(x-1).$$

The leading term is obtained by multiplying the highest-degree terms of each factor:

$$x^3 \cdot x^2 \cdot x = x^6.$$

Thus,

$$f(x) = x^6 + (\text{lower-degree terms}).$$

Since the leading term has even degree and a positive coefficient, it determines the end

behavior.
As $x \to \infty$,

$$f(x) \to \infty.$$

As $x \to -\infty$,

$$f(x) \to \infty.$$

Therefore,

$$\boxed{\lim_{x\to\infty} x^3(x+2)^2(x-1) = \infty}$$

and

$$\boxed{\lim_{x\to-\infty} x^3(x+2)^2(x-1) = \infty.}$$

Thus, both ends of the graph rise upward.

Example: Sketching Using End Behavior

Example 15.19.
Use the limits as $x \to \infty$ and $x \to -\infty$ to sketch a rough graph of

$$y = x^4 - x^6.$$

Solution.
Let

$$f(x) = x^4 - x^6.$$

The leading term is

$$-x^6.$$

Since the leading term has even degree and a negative coefficient,

$$\lim_{x\to\infty} (x^4 - x^6) = -\infty,$$

and

$$\lim_{x\to-\infty} (x^4 - x^6) = -\infty.$$

Thus, both ends of the graph fall downward.
To find the x-intercept(s):

$$x^4 - x^6 = 0,$$

$$x^4(1 - x^2) = 0,$$

so

$$x = -1, \qquad x = 0, \qquad x = 1.$$

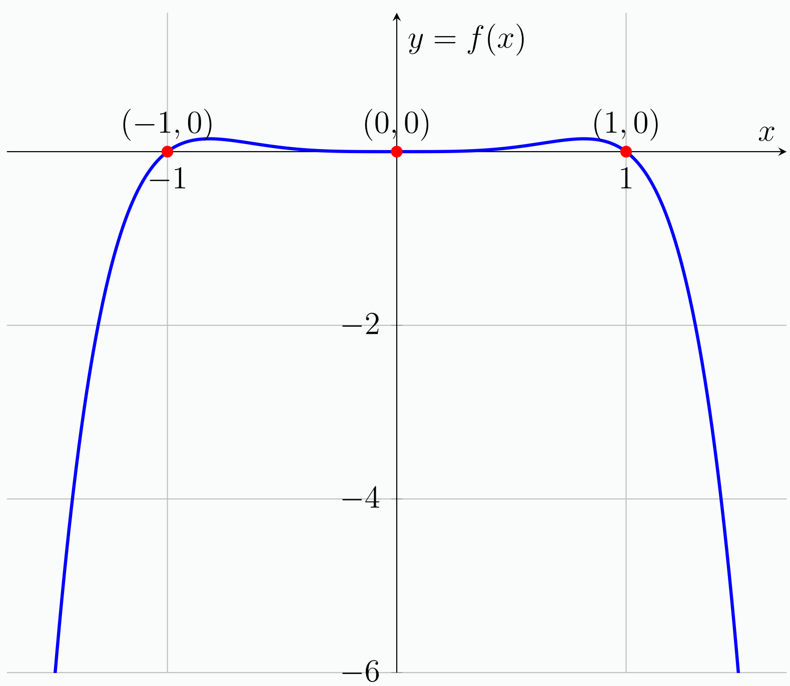
Notice that $x = 0$ is a zero of multiplicity 4, so the graph touches the x-axis at $x = 0$ and turns around.
The zeros at

$$x = \pm 1$$

have multiplicity 1, so the graph crosses the x-axis at those points.
Therefore, the rough sketch has the following features:

- Both ends fall to $-\infty$.
- The graph crosses the x-axis at $x = -1$.
- The graph touches (does not cross) the x-axis at $x = 0$.
- The graph crosses the x-axis again at $x = 1$.
- The graph is symmetric about the y-axis since

$$f(-x) = f(x).$$



Warning

Remember:

- Horizontal asymptotes describe end behavior.
- A graph may cross a horizontal asymptote.
- ∞ is not a number.
- Oscillating functions often fail to have limits.

Exercises

Evaluate the Limits

1.
$$\lim_{x\to\infty} \frac{2x^2-7}{5x^2+x-3}$$

2.
$$\lim_{x\to\infty} \sqrt{\frac{9x^3+8x-4}{x^3-5x+3}}$$

3.
$$\lim_{x\to\infty} \frac{3x-2}{2x+1}$$

4.
$$\lim_{x\to-\infty} \frac{x-2}{x^2+1}$$

5.
$$\lim_{x\to-\infty} \frac{4x^3+6x^2-2}{2x^3-4x+5}$$

6.
$$\lim_{t\to\infty} \frac{\sqrt{t}+t^2}{2t-t^2}$$

7.
$$\lim_{x\to-\infty} \frac{\sqrt{1+4x^6}}{2-x^3}$$

8.
$$\lim_{x\to\infty} \frac{\sqrt{x+3x^2}}{4x-1}$$

9.
$$\lim_{x\to\infty}\left(\sqrt{9x^2+x}-3x\right)$$

10.
$$\lim_{x\to-\infty}\left(\sqrt{4x^2+3x}+2x\right)$$

11.
$$\lim_{x\to\infty}\left(\sqrt{x^2+ax}-\sqrt{x^2+bx}\right)$$

12.
$$\lim_{x\to\infty}\cos x$$

13.
$$\lim_{x\to\infty}(x-\sqrt{x})$$

14.
$$\lim_{x\to\infty}(x^2-x^4)$$

15.
$$\lim_{x\to\infty} x\sin\left(\frac{1}{x}\right)$$

16.
$$\lim_{x\to\infty}\sqrt{x}\sin\left(\frac{1}{x}\right)$$

Horizontal Asymptotes

18.
$$y=\frac{5+4x}{x+3}$$

19.
$$y=\frac{2x^2+1}{3x^2+2x-1}$$

20.
$$y=\frac{2x^2+x-1}{x^2+x-2}$$

21.
$$y=\frac{1+x^4}{x^2-x^4}$$

22.
$$y=\frac{x^3-x}{x^2-6x+5}$$

End Behavior

23. Determine the end behavior of

$$y = 2x^3 - x^4.$$

24. Determine the end behavior of

$$y = x^4 - x^6.$$

25. Determine the end behavior of

$$y = x^3(x+2)^2(x-1).$$

26. Determine the end behavior of

$$y = (3-x)(1+x)^2(1-x)^4.$$

27. Determine the end behavior of

$$y = x^2(x^2-1)^2(x+2).$$

Chapter 16

Curve Sketching

16.1. Summary of Curve Sketching

Brief Description

In this section, we summarize the information gathered throughout Chapter 3 and combine it into a systematic procedure for sketching graphs of functions. Although not every step applies to every function, the following checklist provides a useful framework for producing an accurate sketch.

Important

Curve Sketching Checklist

Given a function $f(x)$:

1. Determine the domain.
2. Find the x- and y-intercepts.
3. Check for symmetry (even, odd, or periodic).
4. Locate vertical, horizontal, and slant asymptotes.
5. Find intervals of increase and decrease using $f'(x)$.
6. Find local maxima and minima.
7. Determine concavity and inflection points using $f''(x)$.
8. Combine all information to sketch the graph.

Step 1: Domain and Intercepts

Domain

The **domain** of a function is the set of all x-values for which the function is defined.

x-intercept(s) & y-intercept

The **y-intercept** occurs at $x = 0$ (if defined), and the **x-intercepts** occur where $f(x) = 0$.

Step 2: Symmetry

Important

Common Symmetries

- If $f(-x) = f(x)$ for all x in the domain, then f is **even** and the graph is symmetric about the y-axis.
- If $f(-x) = -f(x)$ for all x in the domain, then f is **odd** and the graph is symmetric about the origin.
- If $f(x + p) = f(x)$ for all x in the domain, then f is **periodic** with period p.

Step 3: Asymptotes

Vertical Asymptote

The line $x = a$ is a **vertical asymptote** if at least one of the following is true:

$$\lim_{x\to a^-} f(x) = \pm\infty, \qquad \lim_{x\to a^+} f(x) = \pm\infty.$$

Horizontal Asymptote

The line $y = L$ is a **horizontal asymptote** if

$$\lim_{x\to\infty} f(x) = L \quad \text{or} \quad \lim_{x\to-\infty} f(x) = L.$$

Slant Asymptote

The line $y = mx + b$ ($m \neq 0$) is a **slant asymptote** if

$$\lim_{x\to\infty} [f(x) - (mx + b)] = 0$$

or

$$\lim_{x\to-\infty} [f(x) - (mx + b)] = 0.$$

Step 4: Increasing and Decreasing

Important

First Derivative Test

- If $f'(x) > 0$, then f is increasing.
- If $f'(x) < 0$, then f is decreasing.

Step 5: Local Extrema

Critical Number

A **critical number** is a number c in the domain of f such that

$$f'(c) = 0$$

or

$$f'(c) \text{ does not exist.}$$

Important

First Derivative Test for Local Extrema

- If f' changes from positive to negative at c, then $f(c)$ is a local maximum.
- If f' changes from negative to positive at c, then $f(c)$ is a local minimum.

Step 6: Concavity and Inflection Points

Important

Concavity Test

- If $f''(x) > 0$, then the graph is concave upward.
- If $f''(x) < 0$, then the graph is concave downward.
- Inflection points occur where the concavity changes.

16.1.1 Examples

Example 16.1.
Use the curve-sketching procedure to sketch

$$f(x) = x^3 + 3x^2.$$

Solution.

1. **Domain**

   Since f is a polynomial, its domain is

   $$\boxed{(-\infty, \infty).}$$

2. **Intercepts**

   By factoring we get:

   $$f(x) = x^2(x+3).$$

   To find the x-intercepts, set $f(x) = 0$:

   $$x^2(x+3) = 0.$$

   Thus,

   $$x = 0 \qquad \text{or} \qquad x = -3.$$

   Therefore, the x-intercepts are

   $$\boxed{(-3, 0) \text{ and } (0, 0).}$$

Since $x = 0$ has even multiplicity, the graph touches the x-axis at $(0, 0)$ without crossing it.

The y-intercept is also

$$\boxed{(0, 0).}$$

3. **End behavior**

The leading term is x^3. Therefore,

$$\lim_{x \to -\infty} f(x) = -\infty \qquad \text{and} \qquad \lim_{x \to \infty} f(x) = \infty.$$

Thus, the left end falls and the right end rises.

4. **Intervals of increase and decrease**

By differentiating $f(x)$ with respect to x we get:

$$f'(x) = 3x^2 + 6x = 3x(x + 2).$$

Thus, the critical numbers are

$$x = -2 \qquad \text{and} \qquad x = 0.$$

Consider the intervals

$$(-\infty, -2), \qquad (-2, 0), \qquad (0, \infty).$$

Interval	$f'(x)$	Behavior of f
$(-\infty, -2)$	$+$	increasing
$(-2, 0)$	$-$	decreasing
$(0, \infty)$	$+$	increasing

Therefore,

$$\boxed{f \text{ is increasing on } (-\infty, -2) \cup (0, \infty).}$$

$$\boxed{f \text{ is decreasing on } (-2, 0).}$$

5. **Local extrema**

At $x = -2$, f' changes from positive to negative, so f has a local maximum.

$$f(-2) = (-2)^3 + 3(-2)^2 = -8 + 12 = 4.$$

Thus, the local maximum value is

$$\boxed{4}$$

at $x = -2$, and the local maximum point is

$$\boxed{(-2, 4).}$$

At $x = 0$, f' changes from negative to positive, so f has a local minimum.

$$f(0) = 0.$$

Thus, the local minimum value is

$$\boxed{0}$$

at $x = 0$, and the local minimum point is

$$\boxed{(0, 0).}$$

6. **Concavity**

   By differentiating again we get:

$$f''(x) = 6x + 6 = 6(x + 1).$$

   By solving $f''(x) = 0$ for x w get:

$$x = -1.$$

Interval	$f''(x)$	Concavity
$(-\infty, -1)$	$-$	concave down
$(-1, \infty)$	$+$	concave up

   Therefore,

$$\boxed{f \text{ is concave down on } (-\infty, -1).}$$

$$\boxed{f \text{ is concave up on } (-1, \infty).}$$

7. **Inflection point**

   Since the concavity changes at $x = -1$, there is an inflection point there.

$$f(-1) = (-1)^3 + 3(-1)^2 = -1 + 3 = 2.$$

Thus, the inflection point is

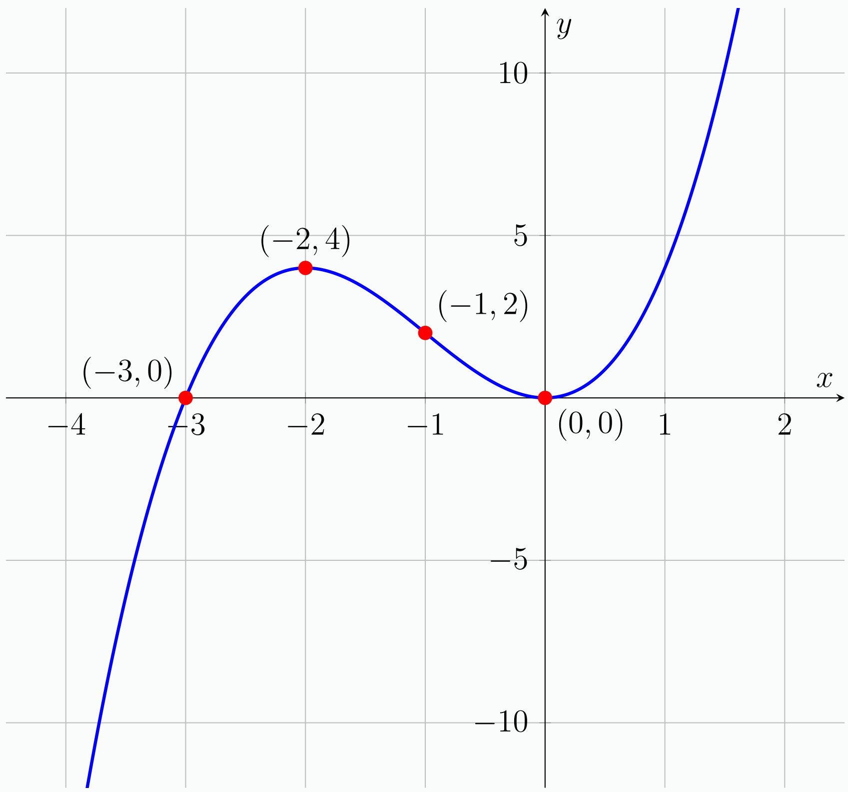
$$\boxed{(-1, 2).}$$

8. **Sketch of the graph**

   The important points are

$$(-3, 0), \qquad (-2, 4), \qquad (-1, 2), \qquad (0, 0).$$

   A sketch is shown below.



Example 16.2.

Use the curve-sketching procedure to sketch

$$f(x) = \frac{x}{x-1}.$$

Solution.

It is useful to rewrite the function as

$$f(x) = 1 + \frac{1}{x-1}.$$

1. **Domain**

   The denominator cannot equal zero:

   $$x - 1 \neq 0.$$

   Thus,

   $$\boxed{\mathrm{Dom}(f) = (-\infty, 1) \cup (1, \infty).}$$

2. **Intercepts**

   To find the x-intercept, set $f(x) = 0$:

   $$\frac{x}{x-1} = 0.$$

   This occurs when the numerator is zero:

   $$x = 0.$$

   But we also have,

   $$f(0) = 0.$$

   Therefore, both the x-intercept and the y-intercept are

   $$\boxed{(0, 0).}$$

3. **Vertical asymptote**

   The denominator is zero at

   $$x = 1.$$

   Since the factor $x - 1$ does not cancel, the vertical asymptote is

   $$\boxed{x = 1.}$$

   The one-sided behavior is

   $$\lim_{x \to 1^-} \frac{x}{x-1} = -\infty \qquad \text{and} \qquad \lim_{x \to 1^+} \frac{x}{x-1} = \infty.$$

4. **Horizontal asymptote and end behavior**

   Since the numerator and denominator have the same degree, the horizontal asymptote is the ratio of the leading coefficients:

$$\boxed{y = 1.}$$

   Equivalently,

$$\lim_{x\to\infty} \frac{x}{x-1} = 1 \qquad \text{and} \qquad \lim_{x\to-\infty} \frac{x}{x-1} = 1.$$

   Because
$$f(x) = 1 + \frac{1}{x-1},$$
   the graph approaches $y = 1$ from above as $x \to \infty$, and from below as $x \to -\infty$.

5. **Intervals of increase and decrease**

   By differentiating $f(x)$ with respect to x we get:

$$f'(x) = \frac{(x-1)(1) - x(1)}{(x-1)^2} = -\frac{1}{(x-1)^2}.$$

   Since
$$f'(x) < 0 \qquad \text{for every} \qquad x \neq 1,$$
   the function is decreasing on each interval of its domain:

$$\boxed{f \text{ is decreasing on } (-\infty, 1) \text{ and } (1, \infty).}$$

   Thus, there are no intervals on which f is increasing.

6. **Local extrema**

   Notice that the derivative is never zero:

$$-\frac{1}{(x-1)^2} \neq 0.$$

   Although $f'(1)$ does not exist, $x = 1$ is not in the domain of f, so it is not a critical number.

   Therefore,

$$\boxed{f \text{ has no local maximum or local minimum values.}}$$

7. **Concavity**

   By differentiating again we get:

   $$f''(x) = \frac{2}{(x-1)^3}.$$

   For $x < 1$, we have

   $$f''(x) < 0.$$

   Thus, f is concave down on

   $$\boxed{(-\infty, 1).}$$

   For $x > 1$, we have

   $$f''(x) > 0,$$

   Thus, f is concave up on

   $$\boxed{(1, \infty).}$$

8. **Inflection points**

   The concavity changes at $x = 1$, but f is not defined there. Therefore, there is no point on the graph at which the concavity changes.

   Thus,

   $$\boxed{f \text{ has no inflection points.}}$$

9. **Range**

   From

   $$y = 1 + \frac{1}{x-1},$$

   the function can never equal 1. Hence,

   $$\boxed{\text{Range}(f) = (-\infty, 1) \cup (1, \infty).}$$

10. **Sketch of the graph**

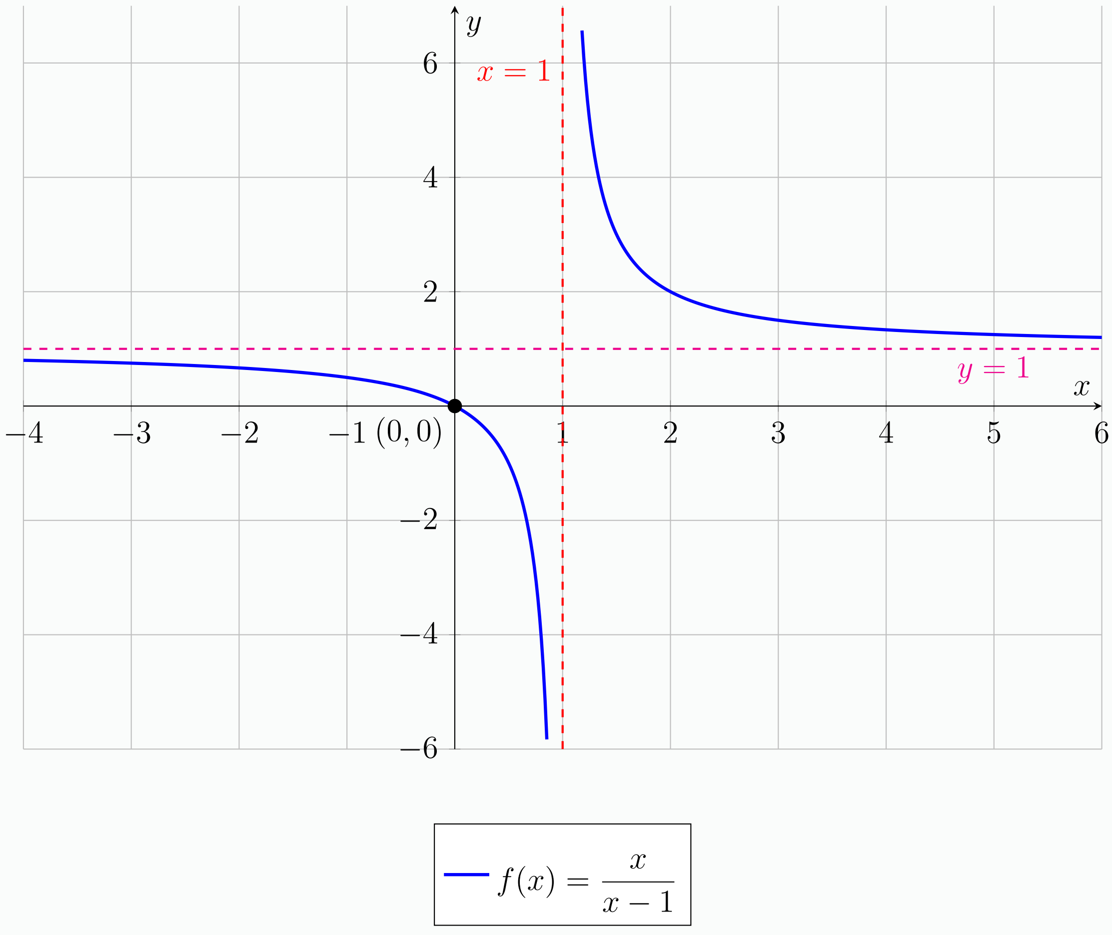
    The important features are:

    $$\text{vertical asymptote: } x = 1,$$

$$\text{horizontal asymptote: } y = 1,$$

$$\text{intercept: } (0, 0).$$

A ketch is given below.



Example 16.3.

Use the curve-sketching procedure to sketch

$$f(x) = \sin x + \sqrt{3}\cos x, \qquad -2\pi \le x \le 2\pi.$$

Solution.

Using the identity

$$a \sin x + b \cos x = R \sin(x + \alpha),$$

where

$$R = \sqrt{1+3} = 2, \qquad \cos\alpha = \frac{1}{2}, \qquad \text{and} \qquad \sin\alpha = \frac{\sqrt{3}}{2},$$

we obtain

$$f(x) = 2\sin\left(x + \frac{\pi}{3}\right).$$

1. **Domain**

$$\boxed{[-2\pi,\ 2\pi]}$$

2. **Intercepts**

   The x-intercepts satisfy

$$2\sin\left(x + \frac{\pi}{3}\right) = 0,$$

   so

$$x + \frac{\pi}{3} = n\pi,$$

   or

$$x = n\pi - \frac{\pi}{3}.$$

   On the interval

$$[-2\pi, 2\pi],$$

   the intercepts are

$$\boxed{-\frac{4\pi}{3},\ -\frac{\pi}{3},\ \frac{2\pi}{3},\ \frac{5\pi}{3}.}$$

   The y-intercept is

$$f(0) = \sqrt{3}.$$

   Thus,

$$\boxed{(0, \sqrt{3}).}$$

3. **End behavior**

   Since the domain is finite,

$$[-2\pi, 2\pi],$$

   there is no end behavior to analyze.

4. **Intervals of increase and decrease**

   By differentiating we get:

$$f'(x) = \cos x - \sqrt{3}\sin x.$$

Equivalently,

$$f'(x) = 2\cos\left(x + \frac{\pi}{3}\right).$$

Critical numbers satisfy

$$\cos\left(x + \frac{\pi}{3}\right) = 0.$$

Thus,

$$x = \frac{\pi}{6} + k\pi.$$

On the interval

$$[-2\pi, 2\pi],$$

these are

$$-\frac{11\pi}{6},\ -\frac{5\pi}{6},\ \frac{\pi}{6},\ \frac{7\pi}{6}.$$

Since

$$f'(x) = 2\cos\left(x + \frac{\pi}{3}\right),$$

the sign chart gives

$$\boxed{\text{Increasing on } \left(-2\pi, -\frac{11\pi}{6}\right) \cup \left(-\frac{5\pi}{6}, \frac{\pi}{6}\right) \cup \left(\frac{7\pi}{6}, 2\pi\right).}$$

$$\boxed{\text{Decreasing on } \left(-\frac{11\pi}{6}, -\frac{5\pi}{6}\right) \cup \left(\frac{\pi}{6}, \frac{7\pi}{6}\right).}$$

5. **Local extrema**

   Evaluate the function:

$$f\left(-\frac{11\pi}{6}\right) = 2,$$

$$f\left(-\frac{5\pi}{6}\right) = -2,$$

$$f\left(\frac{\pi}{6}\right) = 2,$$

$$f\left(\frac{7\pi}{6}\right) = -2.$$

Therefore,

$$\boxed{\text{Local maxima: } \left(-\frac{11\pi}{6}, 2\right),\ \left(\frac{\pi}{6}, 2\right).}$$

$$\boxed{\text{Local minima: } \left(-\frac{5\pi}{6}, -2\right),\ \left(\frac{7\pi}{6}, -2\right).}$$

6. **Concavity**

   By differentiating again we get:

$$f''(x) = -\sin x - \sqrt{3}\cos x = -2\sin\left(x + \frac{\pi}{3}\right).$$

   Hence,

$$f''(x) > 0 \qquad \text{when} \qquad \sin\left(x + \frac{\pi}{3}\right) < 0,$$

   and

$$f''(x) < 0 \qquad \text{when} \qquad \sin\left(x + \frac{\pi}{3}\right) > 0.$$

   Thus,

$$\boxed{\text{Concave up on } \left(-\frac{4\pi}{3}, -\frac{\pi}{3}\right) \cup \left(\frac{2\pi}{3}, \frac{5\pi}{3}\right).}$$

$$\boxed{\text{Concave down on } \left(-2\pi, -\frac{4\pi}{3}\right) \cup \left(-\frac{\pi}{3}, \frac{2\pi}{3}\right) \cup \left(\frac{5\pi}{3}, 2\pi\right).}$$

7. **Inflection points**

   Since

$$f''(x) = 0 \qquad \text{when} \qquad \sin\left(x + \frac{\pi}{3}\right) = 0,$$

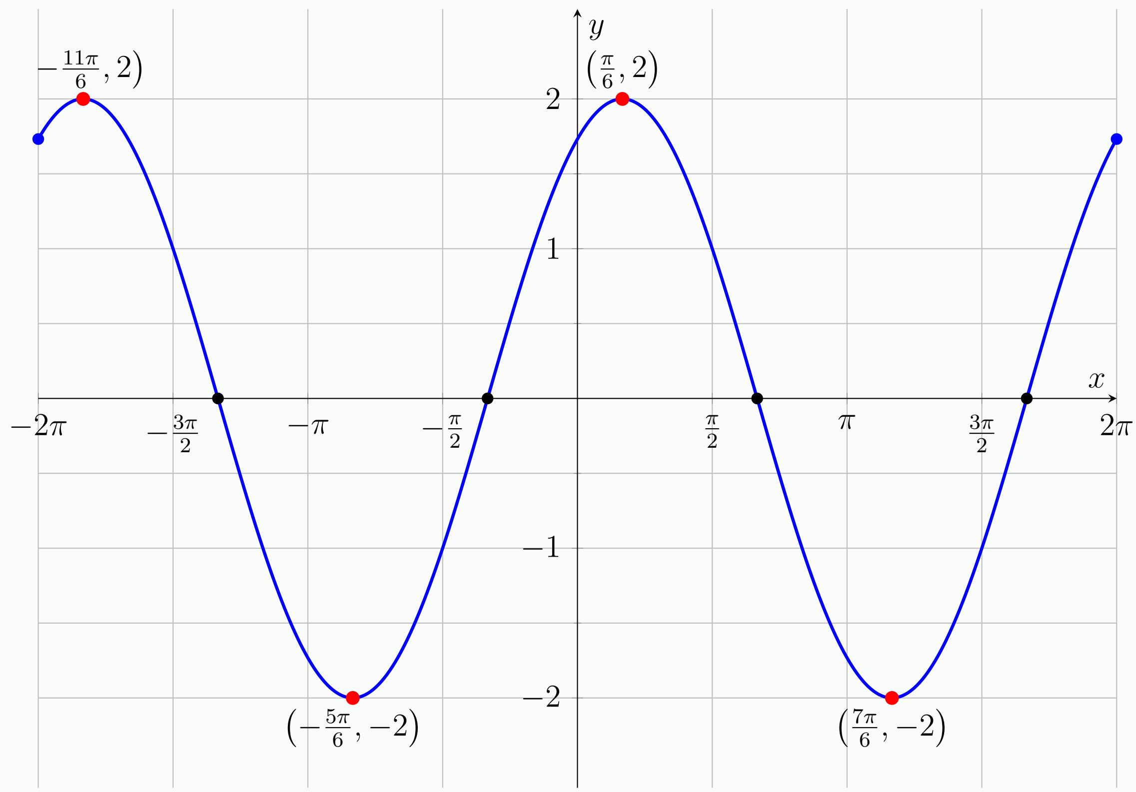
   the inflection points are precisely the x-intercepts:

$$\boxed{\left(-\frac{4\pi}{3}, 0\right),\ \left(-\frac{\pi}{3}, 0\right),\ \left(\frac{2\pi}{3}, 0\right),\ \left(\frac{5\pi}{3}, 0\right).}$$

8. **Sketch**

   The graph is a sine wave with

$$\boxed{\text{Amplitude } 2, \qquad \text{Period } 2\pi, \qquad \text{Phase shift } -\frac{\pi}{3}.}$$



Example 16.4.

Find the slant asymptote of

$$f(x) = \frac{x^2+1}{x+1}.$$

Then use the information to sketch the graph.

Solution.

Consider

$$f(x) = \frac{x^2+1}{x+1}.$$

First, use polynomial division to re-write $f(x)$ as:

$$f(x) = \frac{x^2+1}{x+1} = x - 1 + \frac{2}{x+1}.$$

Because

$$\frac{2}{x+1} \to 0 \qquad \text{as} \qquad x \to \pm\infty,$$

the slant asymptote is

$$\boxed{y = x - 1.}$$

The denominator is zero when

$$x = -1,$$

so the vertical asymptote is

$$\boxed{x = -1.}$$

The behavior near the vertical asymptote is

$$\lim_{x \to -1^-} f(x) = -\infty \qquad \text{and} \qquad \lim_{x \to -1^+} f(x) = \infty.$$

Also,

$$f(x) - (x - 1) = \frac{2}{x+1}.$$

Thus, the graph lies below the slant asymptote when $x < -1$ and above the slant asymptote when $x > -1$.
The function has no x-intercepts because

$$x^2 + 1 > 0$$

for every real number x.
The y-coordinate of the y-intercept is:

$$f(0) = 1.$$

Thus, the graph passes through

$$\boxed{(0, 1).}$$

To obtain additional information for the sketch, by differentiating $f(x)$ with respect to x we get:

$$f'(x) = 1 - \frac{2}{(x+1)^2} = \frac{x^2 + 2x - 1}{(x+1)^2}.$$

By solving $f'(x) = 0$ for x we get:

$$x = -1 \pm \sqrt{2}.$$

By using the Second Derivative Test, one can see that the function has a local maximum at

$$x = -1 - \sqrt{2},$$

with value

$$f(-1-\sqrt{2}) = -2 - 2\sqrt{2}.$$

Thus, the local maximum point is

$$\boxed{\left(-1-\sqrt{2}, -2-2\sqrt{2}\right).}$$

Similarly, the function has a local minimum at

$$x = -1 + \sqrt{2},$$

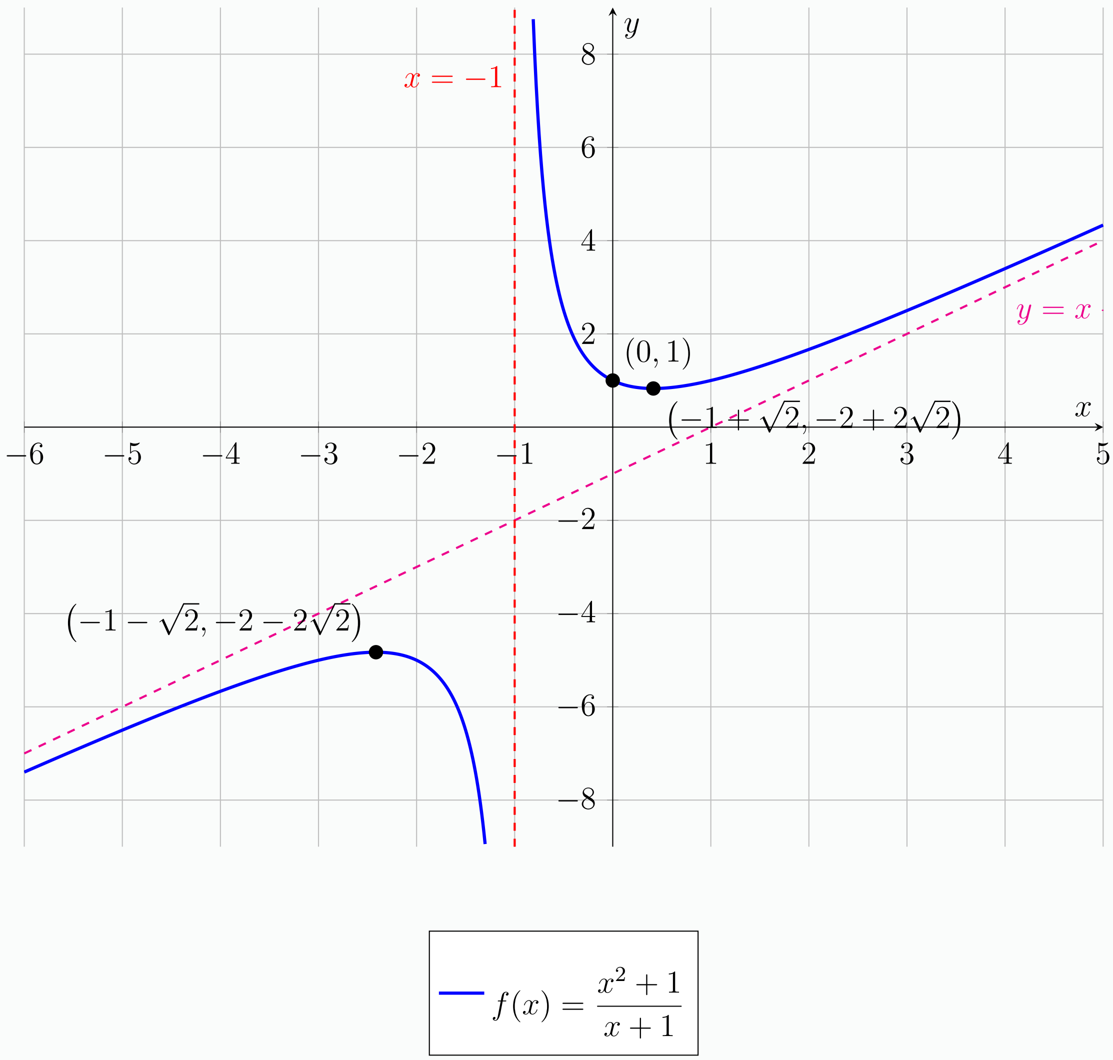
with value

$$f(-1+\sqrt{2}) = -2 + 2\sqrt{2}.$$

Thus, the local minimum point is

$$\boxed{\left(-1+\sqrt{2}, -2+2\sqrt{2}\right).}$$

A sketch is given below.



Example 16.5.
Show that

$$f(x) = \sqrt{4x^2 + 9}$$

has two slant asymptotes and sketch the graph.
Solution.
We will determine its slant asymptotes as $x \to \infty$ and $x \to -\infty$.

1. **As** $x \to \infty$

   The candidate slant asymptote is

$$y = 2x.$$

In fact,

$$\begin{aligned}\lim_{x\to\infty}\left(\sqrt{4x^2+9}-2x\right) &= \lim_{x\to\infty}\frac{(4x^2+9)-4x^2}{\sqrt{4x^2+9}+2x}\\ &= \lim_{x\to\infty}\frac{9}{\sqrt{4x^2+9}+2x}\\ &= 0.\end{aligned}$$

Hence,

$$\boxed{y = 2x}$$

is the slant asymptote as $x \to \infty$.

2. **As** $x \to -\infty$

   Because

$$\sqrt{4x^2} = 2|x| = -2x \qquad \text{for all} \qquad x < 0,$$

   the candidate slant asymptote is

$$y = -2x.$$

   Now we have

$$\begin{aligned}\lim_{x\to-\infty}\left(\sqrt{4x^2+9}+2x\right) &= \lim_{x\to-\infty}\frac{(4x^2+9)-4x^2}{\sqrt{4x^2+9}-2x}\\ &= \lim_{x\to-\infty}\frac{9}{\sqrt{4x^2+9}-2x}\\ &= 0.\end{aligned}$$

   Therefore,

$$\boxed{y = -2x}$$

   is the slant asymptote as $x \to -\infty$.

3. **Additional information for the sketch**

   The function is defined for all real numbers. Thus,

$$\boxed{\text{Domain} = (-\infty, \infty).}$$

Since

$$f(x) \geq 3,$$

the range is

$$\boxed{[3, \infty).}$$

There are no x-intercepts because

$$\sqrt{4x^2 + 9} > 0$$

for every x.

The y-intercept is

$$f(0) = 3,$$

so the graph passes through the point

$$\boxed{(0, 3).}$$

By differentiating we get:

$$f'(x) = \frac{4x}{\sqrt{4x^2 + 9}}.$$

Thus,

$$f'(x) < 0 \quad \text{for all} \quad x < 0 \qquad \text{and} \qquad f'(x) > 0 \quad \text{for all} \quad x > 0.$$

Hence,

$$\boxed{\text{The function decreases on } (-\infty, 0) \text{ and increases on } (0, \infty).}$$

Therefore, the absolute minimum occurs at

$$\boxed{(0, 3).}$$

It is clear that,

$$f''(x) = \frac{36}{(4x^2+9)^{3/2}} > 0 \qquad \text{for all} \qquad x.$$

Thus, the graph is

$$\boxed{\text{concave upward for all } x.}$$

4. **Sketch**

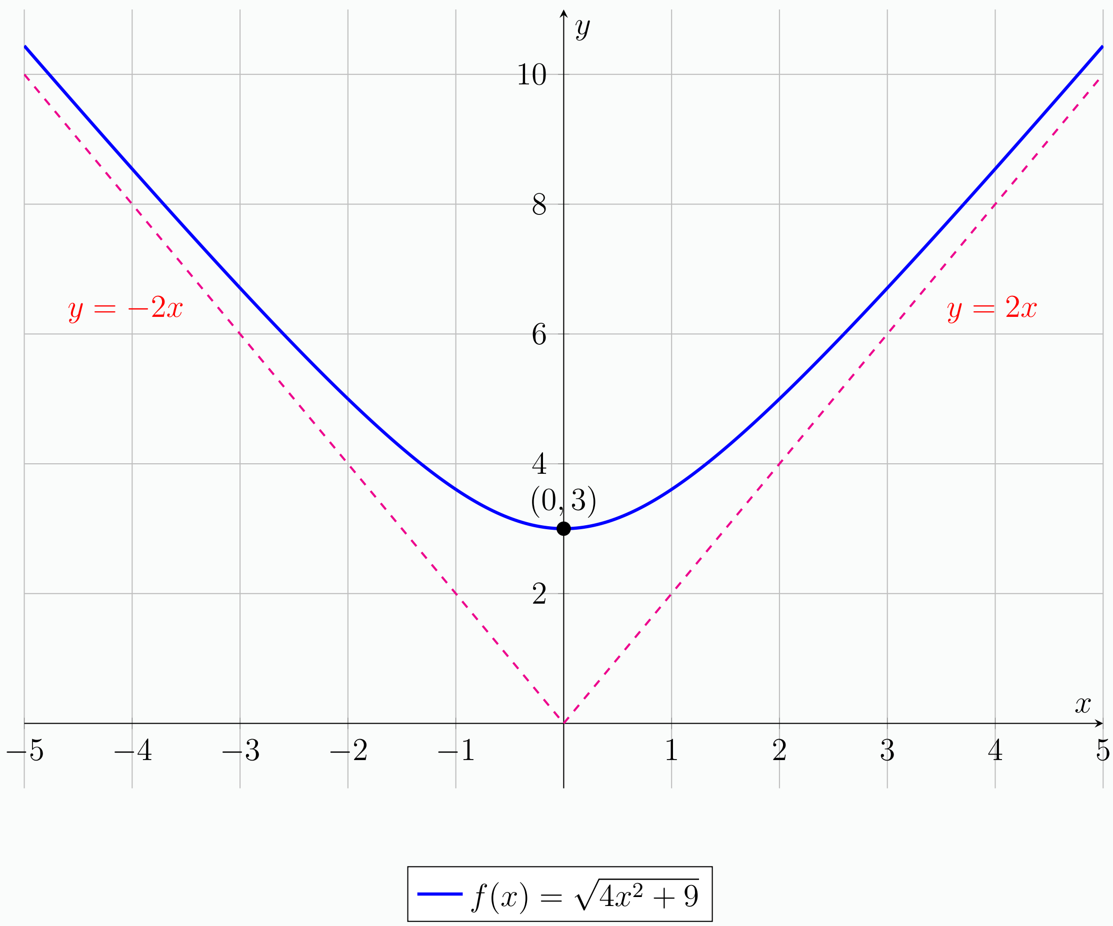
   The graph is symmetric about the y-axis, has minimum point $(0, 3)$, approaches the line

$$y = 2x$$

as $x \to \infty$, and approaches

$$y = -2x$$

as $x \to -\infty$.



Exercises

Use the curve-sketching procedure to sketch each graph.

1. $y = x^3 - 3x$
2. $y = x^4 - 4x$
3. $y = 2 + 3x^2 - x^3$
4. $y = \dfrac{x^2 + 5x}{25 - x^2}$
5. $y = \dfrac{x - x^2}{2 - 3x + x^2}$
6. $y = 1 + \dfrac{1}{x} + \dfrac{1}{x^2}$
7. $y = \dfrac{x}{x^2 - 4}$
8. $y = \dfrac{1}{x^2 - 4}$
9. $y = x + \cos x$
10. $y = x \tan x, \qquad -\dfrac{\pi}{2} < x < \dfrac{\pi}{2}$
11. $y = 2x - \tan x, \qquad -\dfrac{\pi}{2} < x < \dfrac{\pi}{2}$
12. $y = \dfrac{\sin x}{1 + \cos x}$
13. Find the slant asymptote of
$$y = \frac{4x^3 - 10x^2 - 11x + 1}{x^2 - 3x}.$$
14. Find the slant asymptote of
$$y = \frac{2x^3 - 5x^2 + 3x}{x^2 - x - 2}.$$
15. Show that
$$y = \sqrt{x^2 + 4x}$$
has two slant asymptotes and sketch the graph.

Chapter 17

Optimization Problems

17.1. Optimization Problems

Brief Description

Many real-world problems require us to find the largest or smallest possible value of a quantity. Such problems are called **optimization problems**. Typical applications include maximizing area, volume, revenue, and profit, or minimizing cost, distance, and material usage.

Optimization problems are solved using derivatives because maximum and minimum values occur at critical numbers or endpoints of an interval.

A Common Mistake

Many students immediately differentiate the first equation they see.
Instead, always begin by asking:

What quantity am I trying to maximize or minimize?

This quantity is called the objective function.
Only after identifying the objective function should you begin simplifying and differentiating.

17.2. Optimization Strategy

Important

Optimization Strategy

1. Draw a diagram if possible.
2. Introduce variables.
3. Identify the quantity to be maximized or minimized.
4. Write the objective function.
5. Use the given information to express the objective function in one variable.
6. Determine the appropriate domain.
7. Find critical numbers.
8. Determine the maximum or minimum value.
9. Answer the question in the context of the problem.

17.2.1 Objective Functions and Constraints

Objective Function & Constraints

The quantity that must be maximized or minimized is called the **objective function**. Any equation relating the variables is called a **constraint**.

Important

Most optimization problems involve:

- an objective function,
- one or more constraints,
- converting the objective function into a function of a single variable.

17.2.2 Maximizing a Product

Example 17.1.

Find two positive numbers whose sum is 20 and whose product is as large as possible.

Solution.

Let the two positive numbers be

$$x \qquad \text{and} \qquad y.$$

Since their sum is 20, we have

$$x + y = 20.$$

Thus,

$$y = 20 - x.$$

Their product is

$$P = xy.$$

Hence,

$$P(x) = x(20 - x) = 20x - x^2.$$

By differentiating with respect to x we get:

$$P'(x) = 20 - 2x.$$

Solving

$$P'(x) = 0 \quad \text{for} \quad x$$

we get

$$x = 10.$$

Therefore,

$$y = 20 - 10 = 10.$$

Since

$$P''(x) = -2 < 0,$$

the product is maximized at $x = 10$. Hence, the two positive numbers are

$$\boxed{10 \text{ and } 10.}$$

The maximum product is

$$\boxed{100.}$$

17.2.3 Rectangle of Maximum Area

A common optimization problem involves maximizing area.

Example 17.2.
A rectangle has perimeter 100 meters. Find the dimensions of the rectangle having maximum area.

Solution.
Let the length be x meters and the width be y meters. Since the perimeter is 100 meters we have,

$$2x + 2y = 100.$$

Thus,

$$y = 50 - x.$$

The area is

$$A = xy.$$

Therefore,

$$A(x) = x(50 - x) = 50x - x^2.$$

By differentiating with respect to x we get

$$A'(x) = 50 - 2x.$$

By solving

$$A'(x) = 0 \qquad \text{for}$$

we get

$$x = 25.$$

Therefore,

$$y = 50 - 25 = 25.$$

Since

$$A''(x) = -2 < 0,$$

the area is maximized when $x = 25$. Hence, the dimensions of the rectangle are

$$\boxed{25 \text{ m} \times 25 \text{ m.}}$$

The maximum area is

$$\boxed{625 \text{ m}^2.}$$

17.2.4 Open-Top Box

Optimization is frequently used to maximize volume.

Example 17.3.

A square sheet of cardboard is 3 ft by 3 ft. Squares of side length x are cut from the corners and the sides are folded upward to form an open-top box.
Find the dimensions of the box having maximum volume.

Solution.

Let the length be x meters and the width be y meters. Since the perimeter is 100 meters, we have: $2x + 2y = 100$.
Thus,

$$y = 50 - x.$$

The area is

$$A = xy.$$

Therefore,

$$A(x) = x(50 - x) = 50x - x^2.$$

By differentiating with respect to x we get:

$$A'(x) = 50 - 2x.$$

By solving

$$A'(x) = 0 \quad \text{for} \quad x$$

we get

$$x = 25.$$

Therefore,

$$y = 50 - 25 = 25.$$

Because

$$A''(x) = -2 < 0,$$

the area is maximized when $x = 25$.
Hence, the dimensions of the rectangle are

$$\boxed{25 \text{ m} \times 25 \text{ m.}}$$

The maximum area is

$$\boxed{625 \text{ m}^2.}$$

17.2.5 Closest Point Problem

Optimization can also be used to find the point on a curve that is closest to a given point.

Example 17.4.

Find the point on the curve

$$y = \sqrt{x}$$

that is closest to the point $(3, 0)$.

Solution.

Let (x, y) be a generic point on the curve of $y = \sqrt{x}$. Then,

$$(x, y) \equiv (x, \sqrt{x}).$$

Let $L \equiv L(x)$ be the distance between $(x, \sqrt{x})$ and $(3, 0)$.
Then,

$$L = \sqrt{(x-3)^2 + (\sqrt{x} - 0)^2}.$$

That is,

$$L(x) = \sqrt{x^2 - 5x + 9}.$$

By differentiating with respect to x we get

$$L'(x) = \frac{2x - 5}{2\sqrt{x^2 - 5x + 9}}.$$

By solving

$$L'(x) = 0 \quad \text{for} \quad x$$

we get

$$x = \frac{5}{2}.$$

Notice that

$$L'(0) = -\frac{5}{6} < 0 \qquad \text{and} \qquad L'(3) = \frac{1}{2\sqrt{3}} > 0.$$

Because

$$0 < \frac{5}{2} < 3,$$

by using the First Derivative Test, we see that $L(x)$ has a (local) minimum at $x = \dfrac{5}{2}$.
Because $L(x)$ is defined for all real numbers, and

$$\lim_{x \to \pm\infty} L(x) = \infty,$$

we have that
$$L(x) \quad \text{becomes minimum at} \quad x = \frac{5}{2}.$$
Thus, the closet point to the curve $y = \sqrt{x}$ is:
$$\boxed{\left(\frac{5}{2}, \sqrt{\frac{5}{2}}\right).}$$

17.2.6 Largest Rectangle in an Ellipse

Many geometric optimization problems involve maximizing area.

Example 17.5.
Find the area of the largest rectangle that can be inscribed in the ellipse
$$\frac{x^2}{16} + \frac{y^2}{9} = 1.$$
Solution.
Let x be the side length of each square cut from the corners.
After folding, the box has
$$\text{height} = x,$$
and its square base has side length
$$3 - 2x.$$
Therefore, the volume is
$$V(x) = x(3 - 2x)^2, \qquad 0 < x < \frac{3}{2}.$$
By differentiating with respect to x we get:
$$\begin{aligned} V'(x) &= (3 - 2x)^2 - 4x(3 - 2x) \\ &= (3 - 2x)\big((3 - 2x) - 4x\big) \\ &= (3 - 2x)(3 - 6x). \end{aligned}$$
By solving $V'(x) = 0$ for x we get
$$x = \frac{3}{2} \qquad \text{or} \qquad x = \frac{1}{2}.$$
The value $x = \dfrac{3}{2}$ gives a degenerate box with zero volume, so the maximum occurs at

$$x = \frac{1}{2} \text{ ft}.$$

One may use the Second Derivative Test to verify this as

$$V''(x) = 24(x-1),$$

and

$$V''\left(\frac{1}{2}\right) = 24\left(\frac{1}{2}-1\right) = -12 < 0.$$

Thus, the dimensions are

$$3 - 2\left(\frac{1}{2}\right) = 2 \text{ ft}$$

by

$$2 \text{ ft}$$

by

$$\frac{1}{2} \text{ ft}.$$

Therefore, the box with maximum volume has dimensions

$$\boxed{2 \text{ ft} \times 2 \text{ ft} \times \frac{1}{2} \text{ ft}.}$$

Its maximum volume is

$$\boxed{2 \text{ ft}^3.}$$

17.2.7 Farmer's Fencing Problem

A very common application involves maximizing the area subject to a fencing constraint.

Example 17.6.

A farmer has 750 ft of fencing and wishes to enclose a rectangular region divided into four pens by three interior fences parallel to one side.
Find the maximum possible total area.

Solution.
Let x be the length of each of the three interior fences, and let y be the other dimension of the rectangular region.

Since the three interior fences are parallel to the two outer sides of length x, the total fencing consists of five lengths of x and two lengths of y. Therefore,

$$5x + 2y = 750.$$

By solving for y we get:

$$y = \frac{750 - 5x}{2}.$$

The total area is

$$A = xy.$$

Thus,

$$A(x) = x\left(\frac{750 - 5x}{2}\right) = 375x - \frac{5}{2}x^2.$$

By differentiating with respect to x we get:

$$A'(x) = 375 - 5x.$$

By solving $A'(x) = 0$ for x we get:

$$\begin{aligned} 375 - 5x &= 0 \\ x &= \frac{375}{5} \\ x &= 75. \end{aligned}$$

$$x = 75.$$

To find y:

$$y = \frac{750 - 5(75)}{2} = \frac{375}{2} = 187.5.$$

Since

$$A''(x) = -5 < 0,$$

this critical number gives the maximum area.
Therefore, the dimensions of the rectangular region are

$$\boxed{75 \text{ ft} \times 187.5 \text{ ft}.}$$

The maximum possible total area is

$$A_{\text{max}} = 75(187.5) = 14062.5.$$

Hence,

$$\boxed{A_{\text{max}} = 14062.5 \text{ ft}^2.}$$

17.3. Applications to Business and Economics

Optimization is frequently used in economics and business.

Demand | Cost | Revenue | Profit

The **demand function** relates the selling price of a product to the quantity sold:

$$p = p(x),$$

where

- x is the quantity sold,
- $p(x)$ is the price per unit when x units are sold.

In general, demand functions are decreasing because selling more units typically requires lowering the price.

The **cost function** $C(x)$ gives the cost of producing x units.
The **revenue function** is

$$R(x) = x\,p(x),$$

where $p(x)$ is the demand function.
The **profit function** is

$$P(x) = R(x) - C(x).$$

Example 17.7.

A university basketball team found that when ticket prices were \$20, attendance averaged 2,000 fans. When ticket prices were lowered to \$10, attendance increased to 4,000 fans. Assume a linear demand function.

(a). Find the demand function.

(b). Find the revenue function.

Solution.

(a). Using the points

$$(2000, 20) \quad \text{and} \quad (4000, 10),$$

the slope is

$$m = \frac{10 - 20}{4000 - 2000} = -\frac{1}{200}.$$

Using point-slope form:

$$p - 20 = -\frac{1}{200}(x - 2000).$$

By simplifying we get:

$$\boxed{p(x) = 30 - \frac{x}{200}}.$$

(b). The revenue function is

$$R(x) = x\,p(x) = x\left(30 - \frac{x}{200}\right) = 30x - \frac{x^2}{200}.$$

Thus,

$$\boxed{R(x) = 30x - \frac{x^2}{200}}.$$

Important

To maximize profit:

$$P(x) = R(x) - C(x).$$

Find the critical numbers of $P(x)$ and determine which gives the largest profit.
At a profit maximum,

$$P'(x) = 0.$$

Since

$$P'(x) = R'(x) - C'(x),$$

we obtain

$$R'(x) = C'(x).$$

Thus,

$$\boxed{\text{Marginal Revenue} = \text{Marginal Cost}}$$

at the optimal production level.

Example 17.8.

A baseball team found that when ticket prices were \$10, attendance averaged $27,000$.
When ticket prices were lowered to \$8, attendance increased to $33,000$.
Assume a linear demand function.

(a) Find the demand function.

(b) Find the ticket price that maximizes revenue.

Solution.

(a) Let p be the ticket price (in dollars) and let q be the number of tickets sold.

Assume the demand function is linear:

$$q = mp + b.$$

Using the given information,

$$(10, 27000) \qquad \text{and} \qquad (8, 33000),$$

the slope is

$$m = \frac{33000 - 27000}{8 - 10} = \frac{6000}{-2} = -3000.$$

Thus,

$$q = -3000p + b.$$

Using the point $(10, 27000)$,

$$27000 = -3000(10) + b,$$

so

$$b = 57000.$$

Hence the demand function is

$$\boxed{q = 57000 - 3000p.}$$

(b) Revenue is

$$R = pq.$$

Substitute the demand function:

$$R(p) = p(57000 - 3000p) = -3000p^2 + 57000p.$$

By differentiating we get:

$$R'(p) = -6000p + 57000.$$

Set the derivative equal to zero:

$$-6000p + 57000 = 0,$$

so

$$p = \frac{57000}{6000} = 9.5.$$

Since

$$R''(p) = -6000 < 0,$$

the revenue is maximized at

$$\boxed{p = \$9.50.}$$

The corresponding attendance is

$$q = 57000 - 3000(9.5) = 28500.$$

The maximum revenue is

$$R = (9.5)(28500) = 270750.$$

Thus,

$$\boxed{\text{Maximum revenue} = \$270{,}750.}$$

$$\boxed{\begin{array}{l} \text{(a) } q = 57000 - 3000p, \\ \text{(b) Optimal ticket price } = \$9.50. \end{array}}$$

Exercises

1. Find two positive numbers whose sum is 30 and whose product is as large as possible.

2. Find two positive numbers whose product is 144 and whose sum is as small as possible.

3. A rectangle has a perimeter of 80 meters. Find the dimensions that maximize its area.

4. A rectangle has an area of 200 m^2. Find the dimensions that minimize its perimeter.

5. A farmer has 400 ft of fencing and wishes to enclose a rectangular field. Find the dimensions that maximize the enclosed area.

6. A rectangular garden is to have an area of 1200 ft^2. Find the dimensions that require the least amount of fencing.

7. A square sheet of cardboard measuring 12 in by 12 in is used to make an open-top box by cutting equal squares from the corners and folding up the sides. Find the dimensions of the box having maximum volume.

8. A rectangular sheet of cardboard measuring 16 in by 10 in is used to make an open-top box by cutting equal squares from the corners. Find the dimensions of the box having maximum volume.

9. A square sheet of cardboard has a side length of 20 cm. Squares of side length x are cut from each corner and folded upward. Write a formula for the volume of the resulting box and determine the domain of the function.

10. A box with a square base and open top is to have a volume of 500 cm^3. Express the surface area as a function of the side length of the base.

11. A rectangle is inscribed under the parabola

$$y = 12 - x^2$$

with its base on the x-axis. Express the area of the rectangle as a function of x.

12. A rectangle is inscribed in the semicircle

$$x^2 + y^2 = 25, \qquad y \geq 0.$$

Express the area of the rectangle as a function of x.

Chapter 18

Antiderivatives

18.1. Antiderivatives

Brief Description

In this section we study the process of **antidifferentiation**, which is the reverse of differentiation. Given a function f, we seek a function F whose derivative is f.

Antiderivatives arise naturally in many applications. For example,

- If velocity is known, antidifferentiation gives position.
- If acceleration is known, antidifferentiation gives velocity.
- Antiderivatives form the foundation of integral calculus.

The main goal of this section is to learn how to recognize antiderivatives and compute them using basic formulas.

Definition

A function F is called an **antiderivative** of f on an interval I if

$$F'(x) = f(x)$$

for all $x \in I$.

For example, we have

$$\frac{d}{dx}\left(\frac{x^3}{3}\right) = x^2.$$

Therefore,

$$F(x) = \frac{x^3}{3}$$

is an antiderivative of

$$f(x) = x^2.$$

However,

$$G(x) = \frac{x^3}{3} + 5$$

also satisfies

$$G'(x) = x^2.$$

In fact,

$$\frac{x^3}{3} + C$$

is an antiderivative of x^2 for every constant C.

Important

Different antiderivatives of the same function differ only by a constant.

18.1.1 The General Antiderivative

Theorem 18.1 (General Antiderivative)

Suppose F is an antiderivative of f on an interval I.
Then every antiderivative of f is of the form

$$F(x) + C,$$

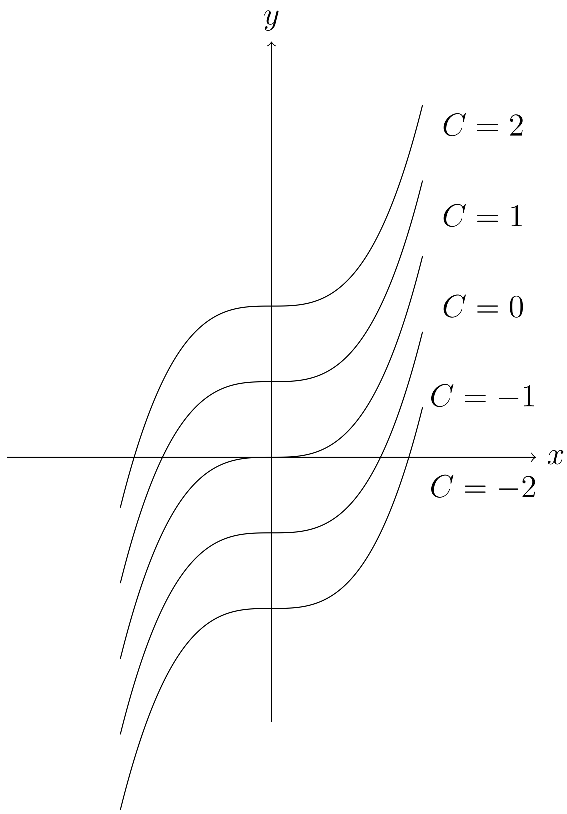
where C is an arbitrary constant.
This collection of all antiderivatives is called the **general antiderivative** of f.

18.1.2 A Family of Antiderivatives

The graphs of

$$y = \frac{x^3}{3} + C$$

all have the same derivative x^2 and differ only by vertical shifts.



18.2. Basic Antiderivative Formulas

$f(x)$	General Antiderivative
k	$kx + C$
$x^n,\ n \neq -1$	$\dfrac{x^{n+1}}{n+1} + C$
$\cos x$	$\sin x + C$
$\sin x$	$-\cos x + C$
$\sec^2 x$	$\tan x + C$
$\sec x \tan x$	$\sec x + C$

18.3. Properties of Antiderivatives

If $F'(x) = f(x)$ and $G'(x) = g(x)$, then

$$\int [f(x) + g(x)]\, dx = \int f(x)\, dx + \int g(x)\, dx.$$

Also,

$$\int cf(x)\, dx = c \int f(x)\, dx.$$

These rules allow us to antidifferentiate complicated expressions term-by-term.

Example 18.1.

Find the most general antiderivative of

$$f(x) = 4x + 7.$$

Solution.

$$\begin{aligned} \int f(x)\ dx &= \int (4x + 7)\ dx \\ &= 4 \cdot \frac{x^2}{2} + 7x + C \\ &= 2x^2 + 7x + C. \end{aligned}$$

$$\boxed{2x^2 + 7x + C.}$$

Example 18.2.

Find the most general antiderivative of

$$f(x) = 3\sqrt{x} - 2\sqrt[3]{x}.$$

Solution.

We have

$$f(x) = 3x^{\frac{1}{2}} - 2x^{\frac{1}{3}}.$$

Thus,

$$\int f(x)\ dx = \int (3x^{\frac{1}{2}} - 2x^{\frac{1}{3}})\ dx$$

$$\begin{aligned}
&= 3 \cdot \frac{x^{\frac{1}{2}+1}}{\frac{1}{2}+1} - 2 \cdot \frac{x^{\frac{1}{3}+1}}{\frac{1}{3}+1} + C \\
&= 2x^{\frac{3}{2}} - \frac{3}{2}x^{\frac{4}{3}} + C.
\end{aligned}$$

$$\boxed{2x^{\frac{3}{2}} - \frac{3}{2}x^{\frac{4}{3}} + C.}$$

Example 18.3.
Find the most general antiderivative of

$$f(x) = 3\cos x - 4\sin x.$$

Solution.

$$\begin{aligned}
\int f(x)\ dx &= \int (3\cos x - 4\sin x)\ dx \\
&= 3\sin x - 4(-\cos x) + C \\
&= 3\sin x + 4\cos x + C.
\end{aligned}$$

$$\boxed{3\sin x + 4\cos x + C}$$

18.4. Initial Value Problems and Finding Functions

18.4.1 Initial Value Problems

Often we are given a derivative and additional information about the function itself. The additional information is used to determine the constant of integration.

For example, suppose we want to find $f(x)$ such that

$$f'(x) = 3x^2 \quad \text{and} \quad f(1) = 5.$$

Because $f'(x) = 3x^2$, we get

$$f(x) = x^3 + C.$$

Because $f(1) = 5$, we get $5 = 1^3 + C$. This implies

$$C = 4.$$

Therefore,

$$f(x) = x^3 + 4.$$

Initial Value Problems

An **initial value problem** consists of finding a function satisfying a differential equation together with one or more conditions on the function.

Important

To solve an initial value problem:

1. Find the general antiderivative.
2. Use the given condition(s).
3. Solve for the constant(s).
4. Write the particular solution.

18.4.2 Finding a Function from Its First Derivative

Example 18.4.

Find the function f if

$$f'(x) = 5x^4 - 3x^2 + 4, \qquad f(-1) = 2.$$

Solution.

We have

$$\begin{aligned} f(x) &= \int f'(x)\ dx \\ &= \int (5x^4 - 3x^2 + 4)\ dx \\ &= x^5 - x^3 + 4x + C. \end{aligned}$$

Because $f(-1) = 2$, we have

$$(-1)^5 - (-1)^3 + 4(-1) + C = 2.$$

By simplifying we then get

$$C = 6.$$

Therefore,

$$\boxed{f(x) = x^5 - x^3 + 4x + 6.}$$

18.4.3 Finding a Function from Its Second Derivative

If the second derivative is known, then we antidifferentiate it twice.

$$f''(x) \longrightarrow f'(x) \longrightarrow f(x)$$

Each antidifferentiation introduces a new constant.

Example 18.5.
Find the function f if

$$f''(\theta) = \sin\theta + \cos\theta \quad \text{and} \quad f(0) = 3, f'(0) = 4.$$

Solution.
We have

$$\begin{aligned} f'(\theta) &= \int f'(\theta)\ d\theta \\ &= \int (\sin\theta + \cos\theta)\ d\theta \\ &= -\cos\theta + \sin\theta + C. \end{aligned}$$

Because $f'(0) = 4$, we get

$$\begin{aligned} -\cos 0 + \sin 0 + C &= 4 \\ -1 + C &= 4 \\ C &= 5. \end{aligned}$$

Thus,

$$f'(\theta) = -\cos\theta + \sin\theta + 5.$$

Then,

$$f(\theta) = \int f'(\theta)\ d\theta$$

$$= \int (-\cos\theta + \sin\theta + 5)\ d\theta$$
$$= -\sin\theta - \cos\theta + 5\theta + D.$$

Because $f(0) = 4$,

$$\begin{aligned} -\sin 0 - \cos 0 + 5(0) + D &= 4 \\ -1 + D &= 4 \\ D &= 5. \end{aligned}$$

Thus,

$$\boxed{f(\theta) = -\sin\theta - \cos\theta + 5\theta + 5.}$$

18.4.4 Graph Interpretation

Recall:

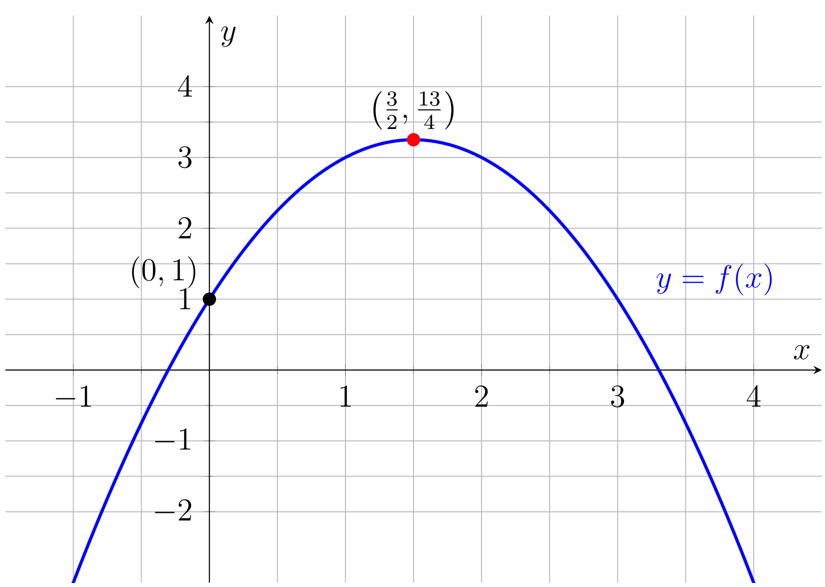
$$f'(x) = \text{slope of } f.$$

Therefore, if we know $f'(x)$, we can recover $f(x)$ through antidifferentiation.

For example, if

$$f'(x) = 3 - 2x,$$

then f must be a quadratic function.



A derivative determines a family of functions that differ only by constants.

18.4.5 Applications

Sometimes, a derivative is given as the slope of a tangent line.

Example 18.6.
The graph of f passes through the point

$$(2, 5),$$

and the slope of the tangent line at $(x, f(x))$ is

$$3 - 4x.$$

Find $f(1)$.
Solution.
The slope of the tangent line is

$$f'(x) = 3 - 4x.$$

Integrate to obtain $f(x)$:

$$f(x) = \int (3 - 4x)\, dx = 3x - 2x^2 + C.$$

Because the point $(2, 5)$ is on the curve we get

$$5 = 3(2) - 2(2)^2 + C,$$

$$5 = 6 - 8 + C,$$

$$C = 7.$$

Hence,

$$f(x) = 3x - 2x^2 + 7.$$

Now evaluate $f(1)$:

$$f(1) = 3(1) - 2(1)^2 + 7 = 3 - 2 + 7 = 8.$$

Therefore,

$$\boxed{f(1) = 8.}$$

18.5. Rectilinear Motion Applications

18.5.1 Rectilinear Motion

One of the most important applications of antiderivatives is the study of motion along a straight line.

Suppose an object moves along a coordinate axis, and its position at time t is given by

$$s = s(t).$$

Then

$$v(t) = s'(t)$$

is called the **velocity** and

$$a(t) = v'(t) = s''(t)$$

is called the **acceleration**.

$$a(t) \quad \underset{\text{Antidifferentiate}}{\longrightarrow} \quad v(t) \quad \underset{\text{Antidifferentiate}}{\longrightarrow} \quad s(t)$$

Important

To find position from acceleration:

1. Antidifferentiate acceleration to obtain velocity.
2. Use any initial velocity condition.
3. Antidifferentiate velocity to obtain position.
4. Use any initial position condition.

18.5.2 Position and Velocity

Example 18.7.

A particle has velocity

$$v(t) = \sin t - \cos t$$

and satisfies

$$s(0) = 0.$$

Find the position function.

Solution.

Since velocity is the derivative of position,

$$s'(t) = v(t) = \sin t - \cos t.$$

By integrating we get:

$$\begin{aligned} s(t) &= \int (\sin t - \cos t)\, dt \\ &= -\cos t - \sin t + C. \end{aligned}$$

Because $s(0) = 0$,

$$0 = -\cos 0 - \sin 0 + C,$$

$$0 = -1 - 0 + C.$$

Thus,

$$C = 1.$$

Therefore, the position function is

$$\boxed{s(t) = 1 - \cos t - \sin t.}$$

18.5.3 Acceleration and Position

Example 18.8.

A particle has acceleration

$$a(t) = 2t + 1$$

and satisfies

$$s(0) = 3, \qquad v(0) = -2.$$

Find the position function.

Solution.

Since

$$a(t) = v'(t) = 2t + 1,$$

integrate to find the velocity:

$$\begin{aligned} v(t) &= \int (2t + 1)\, dt \\ &= t^2 + t + C. \end{aligned}$$

Because $v(0) = -2$, we get:

$$-2 = 0 + 0 + C,$$

so,

$$C = -2.$$

Hence,

$$v(t) = t^2 + t - 2.$$

Now integrate the velocity to obtain the position:

$$\begin{aligned} s(t) &= \int (t^2 + t - 2)\, dt \\ &= \frac{t^3}{3} + \frac{t^2}{2} - 2t + C. \end{aligned}$$

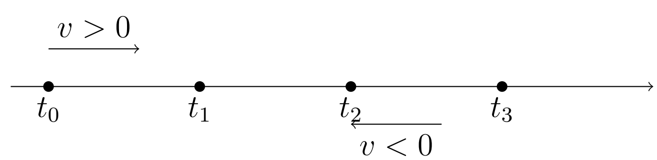
Because $s(0) = 3$, we get:

$$3 = C.$$

Therefore, the position function is

$$\boxed{s(t) = \frac{t^3}{3} + \frac{t^2}{2} - 2t + 3.}$$

18.5.4 Motion Diagram



Positive velocity corresponds to motion to the right.

Negative velocity corresponds to motion to the left.

18.5.5 Falling Objects

Near the surface of the earth, the acceleration due to gravity is approximately

$$a(t) = -32 \qquad \text{ft/s}^2$$

or

$$a(t) = -9.8 \qquad \text{m/s}^2.$$

Antidifferentiating gives

$$v(t) = -32t + v_0$$

and

$$s(t) = -16t^2 + v_0 t + s_0.$$

Important

For motion measured in feet,

$$s(t) = -16t^2 + v_0 t + s_0.$$

For motion measured in meters,

$$s(t) = -4.9t^2 + v_0 t + s_0.$$

18.5.6 Dropped Object

Example 18.9.

A stone is dropped from a height of

$$450 \text{ m}.$$

Find the height above the ground after t seconds. You may assume that acceleration due to gravity is $g = 10 \text{ ms}^{-2}$.

Solution.

Since the stone is dropped, its initial velocity is

$$v(0) = 0.$$

The acceleration due to gravity is

$$a(t) = -10 \text{ m/s}^2.$$

Integrate to find the velocity:

$$\begin{aligned} v(t) &= \int -10\,dt \\ &= -10t + C. \end{aligned}$$

Using $v(0) = 0$, we get

$$C = 0,$$

so,

$$v(t) = -10t.$$

Now integrate the velocity to obtain the height:

$$\begin{aligned} s(t) &= \int -10t\,dt \\ &= -5t^2 + D. \end{aligned}$$

Since the stone starts at a height of 450 m,

$$s(0) = 450,$$

so,

$$D = 450.$$

Therefore, the height of the stone after t seconds is

$$\boxed{s(t) = 450 - 5t^2.}$$

This formula is valid until the stone reaches the ground, which occurs when

$$450 - 5t^2 = 0 \quad \Longrightarrow \quad t = \sqrt{90} = 3\sqrt{10} \approx 9.49 \text{ s}.$$

18.5.7 Thrown Downward

Example 18.10.

A stone is thrown downward from a height of

$$450 \text{ m}$$

with an initial speed of

$$5 \text{ m/s}.$$

Find a formula for the height of the stone after t seconds. You may assume that acceleration due to gravity is $g = 10$ ms^{-2}.

Solution.

Take upward to be the positive direction.

Since the stone is thrown downward,

$$v(0) = -5 \text{ m/s}.$$

The acceleration due to gravity is

$$a(t) = -10 \text{ m/s}^2.$$

By integrating to find the velocity, we gwt:

$$\begin{aligned} v(t) &= \int -10\, dt \\ &= -10t + C. \end{aligned}$$

Because $v(0) = -5$,

$$-5 = C,$$

so,

$$v(t) = -10t - 5.$$

Now integrate the velocity to obtain the height:

$$\begin{aligned} s(t) &= \int (-10t - 5)\, dt \\ &= -5t^2 - 5t + D. \end{aligned}$$

Since the stone starts at a height of 450 m,

$$s(0) = 450,$$

so,

$$D = 450.$$

Therefore, the height of the stone after t seconds is

$$\boxed{s(t) = 450 - 5t^2 - 5t.}$$

This formula is valid until the stone reaches the ground.

18.5.8 Constant Acceleration Formula

For motion in a straight line with constant acceleration a,

$$v(t) = at + v_0.$$

$$s(t) = \frac{1}{2}at^2 + v_0 t + s_0.$$

Here

- a = acceleration,
- v_0 = initial velocity,
- s_0 = initial position.

Exercises

Practice Problems A

Find the most general antiderivative.

1. $f(x) = x^2 - 3x + 2$
2. $f(x) = 6x^5 - 8x^4 - 9x^2$
3. $f(x) = \sqrt{2}$
4. $f(x) = \pi^2$
5. $f(x) = \dfrac{10}{x^9}$
6. $f(x) = \dfrac{5 - 4x^3 + 2x^6}{x^6}$
7. $g(t) = \dfrac{1 + t + t^2}{\sqrt{t}}$
8. $h(\theta) = 2\sin\theta - \sec^2\theta$
9. $g(v) = 5 + 3\sec^2 v$
10. $f(t) = 8\sqrt{t} - \sec t \tan t$
11. $f(x) = 1 + 2\sin x + \dfrac{3}{\sqrt{x}}$
12. $f(x) = x(12x + 8)$
13. $f(x) = 4 - \sqrt[3]{x}$
14. $f(x) = x^{2/3} + x^{-2/3}$
15. $f(t) = 12 + \sin t$

Practice Problems B

Find the function f.

1. $$f''(x) = 20x^3 - 12x^2 + 6x$$
2. $$f''(x) = x^6 - 4x^4 + x + 1$$
3. $$f''(x) = 4 - \sqrt[3]{x}$$
4. $$f''(x) = x^{2/3} + x^{-2/3}$$
5. $$f'''(t) = 12 + \sin t$$
6. $$f'(x) = 1 + 3\sqrt{x}, \qquad f(4) = 25$$

7.
$$f'(x) = 5x^4 - 3x^2 + 4, \qquad f(-1) = 2$$

8.
$$f'(x) = \sqrt{x}(6 + 5x), \qquad f(1) = 10$$

9.
$$f'(t) = t + \frac{1}{t^3}, \qquad t > 0, \qquad f(1) = 6$$

10.
$$f'(t) = \sec t(\sec t + \tan t), \qquad -\frac{\pi}{2} < t < \frac{\pi}{2}, \qquad f\left(\frac{\pi}{4}\right) = -1$$

11.
$$f'(x) = \frac{x+1}{\sqrt{x}}, \qquad f(1) = 5$$

12.
$$f''(x) = -2 + 12x - 12x^2, \qquad f(0) = 4, \qquad f'(0) = 12$$

13.
$$f''(x) = 8x^3 + 5, \qquad f(1) = 0, \qquad f'(1) = 8$$

14.
$$f''(\theta) = \sin\theta + \cos\theta, \qquad f(0) = 3, \qquad f'(0) = 4$$

15.
$$f''(t) = 4 - \frac{6}{t^4}, \qquad f(1) = 6, \qquad f'(2) = 9$$

16.
$$f''(x) = 4 + 6x + 24x^2, \qquad f(0) = 3, \qquad f(1) = 10$$

17.
$$f''(x) = 20x^3 + 12x^2 + 4, \qquad f(0) = 8, \qquad f(1) = 5$$

18. The graph of f passes through $(2, 5)$ and has a derivative
$$f'(x) = 3 - 4x.$$
Find $f(1)$.

19. Find a function f such that
$$f'(x) = x^3$$

and the line

$$x + y = 0$$

is tangent to the graph of f.

Practice Problems C

Find the position function.

1. $$v(t) = \sin t - \cos t, \qquad s(0) = 0.$$

2. $$v(t) = t^2 - 3\sqrt{t}, \qquad s(4) = 8.$$

3. $$a(t) = 2t + 1, \qquad s(0) = 3, \qquad v(0) = -2.$$

4. $$a(t) = 3\cos t - 2\sin t, \qquad s(0) = 0, \qquad v(0) = 4.$$

5. $$a(t) = 10\sin t + 3\cos t, \qquad s(0) = 0, \qquad s(2\pi) = 12.$$

6. $$a(t) = t^2 - 4t + 6, \qquad s(0) = 0, \qquad s(1) = 20.$$

A stone is dropped from a height of 450 m.

7. Find the height after t seconds.
8. How long does it take to reach the ground?
9. What is its velocity when it strikes the ground?
10. If the stone is thrown downward at 5 m/s, how long does it take to hit the ground?

Additional Applications

11. Two balls are thrown upward from a cliff. The first is thrown upward at 48 ft/s. One second later a second ball is thrown upward at 24 ft/s. Do the balls ever pass each other?

12. A stone is dropped from a cliff and strikes the ground at a speed of 120 ft/s. Find the height of the cliff.

13. What constant acceleration is required to increase the speed of a car from 30 mi/h to 50 mi/h in 5 seconds?

14. A car brakes with a constant deceleration of 16 ft/s^2 and leaves skid marks 200 ft long. How fast was the car traveling when the brakes were applied?

15. A car is traveling at 100 km/h and must stop within 80 m. What constant deceleration is required?

16. A high-speed train accelerates and decelerates at 4 ft/s^2. Its maximum cruising speed is 90 mi/h.

    (a) What is the maximum distance traveled if it cruises for 15 minutes?

    (b) If the train must start and stop within 15 minutes, what is the maximum distance traveled?

    (c) What is the minimum time required to travel 45 miles?

Chapter 19

Areas and Distances

19.1. Areas and Distances

Brief Description

In this section, we investigate one of the central problems that led to the development of calculus: finding the area of a region under a curve.

We begin by approximating areas using rectangles. As the rectangles become thinner and more numerous, the approximations become increasingly accurate. This idea leads naturally to the concept of a definite integral, which will be studied in later sections.

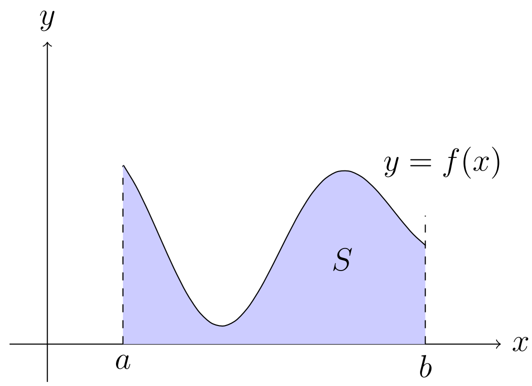
We also examine a related problem involving distance traveled by a moving object. Surprisingly, both problems are connected through the same geometric idea: area under a curve.

19.2. The Area Problem

Suppose f is a continuous function satisfying $f(x) \geq 0$ on the interval $[a, b]$.

We would like to find the area of the region bounded by

$$y = f(x), \qquad x = a, \qquad x = b, \qquad \text{and the } x\text{-axis.}$$



The region shown above is

$$S = \{(x, y) \mid a \leq x \leq b,\ 0 \leq y \leq f(x)\}.$$

Our goal is to determine the area of S.

19.2.1 Approximating Area with Rectangles

One approach is to divide the interval $[a, b]$ into n equal pieces.

The width of each rectangle is

$$\Delta x = \frac{b - a}{n}.$$

Choosing a sample point x_i in each subinterval, we construct a rectangle of height $f(x_i)$.

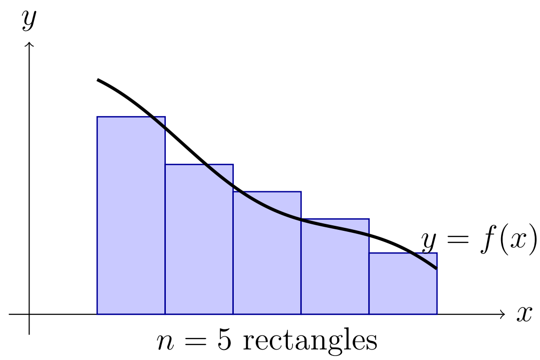
The area of the ith rectangle is approximately

$$f(x_i)\Delta x.$$

Adding the areas of all rectangles gives the approximation

$$R_n = f(x_1)\Delta x + f(x_2)\Delta x + \cdots + f(x_n)\Delta x.$$

This sum is called a **Riemann Sum**.



As n increases, the rectangles provide a better approximation to the actual area.

Definition

The **area under the curve** $y = f(x)$ from $x = a$ to $x = b$ is defined as the limit of the corresponding Riemann sums:

$$A = \lim_{n\to\infty} \left[f(x_1)\Delta x + f(x_2)\Delta x + \cdots + f(x_n)\Delta x \right].$$

Important

The basic idea is simple:

$$\text{Area} \approx \text{sum of rectangle areas.}$$

Using more rectangles generally produces a more accurate approximation.

19.2.2 Left and Right Endpoint Approximations

Two common choices for the sample points are:

- Left endpoints
- Right endpoints

These lead to the **Left Riemann Sum** and the **Right Riemann Sum**.

For a decreasing function:

$$L_n = \text{overestimate}, \qquad R_n = \text{underestimate}.$$

For an increasing function:

$$L_n = \text{underestimate}, \qquad R_n = \text{overestimate}.$$

Worked Examples

Example 19.1.
Estimate the area under

$$f(x) = x^2$$

on the interval $[0, 2]$ using two rectangles and right endpoints.
Solution.
Since $n = 2$,

$$\Delta x = \frac{2-0}{2} = 1.$$

The right endpoints are

$$x_1 = 1, \qquad x_2 = 2.$$

Evaluate the function:

$$f(1) = 1, \qquad f(2) = 4.$$

Therefore,

$$R_2 = f(1)\Delta x + f(2)\Delta x = 1(1) + 4(1) = 5.$$

Thus the estimated area is

$$\boxed{5}.$$

Since $f(x) = x^2$ is increasing on $[0, 2]$, the right-endpoint approximation is an overestimate.

Example 19.2.
Estimate the area under

$$f(x) = 4 - x$$

on the interval $[0, 4]$ using four rectangles and left endpoints.
Solution.
Since

$$\Delta x = \frac{4-0}{4} = 1,$$

the left endpoints are

$$0,\ 1,\ 2,\ 3.$$

Evaluate the function:

$$f(0) = 4, \qquad f(1) = 3, \qquad f(2) = 2, \qquad f(3) = 1.$$

Therefore,

$$L_4 = (4+3+2+1)(1) = 10.$$

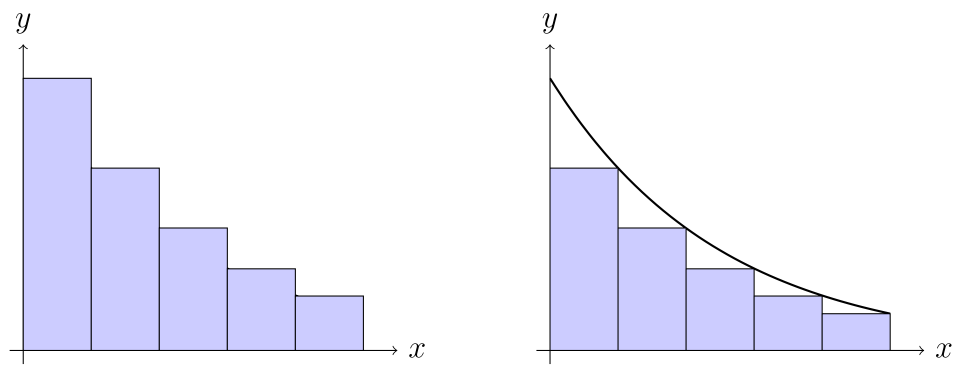
Hence,

$$\boxed{L_4 = 10}.$$

Since $f(x) = 4 - x$ is decreasing, the left-endpoint approximation is an overestimate.

19.2.3 Visualizing Left and Right Sums

For decreasing functions, left rectangles lie mostly above the curve, while right rectangles lie mostly below the curve.



Left Endpoint Approximation   Right Endpoint Approximation

Important

For an increasing function:

$$L_n < \text{Actual Area} < R_n.$$

For a decreasing function:

$$R_n < \text{Actual Area} < L_n.$$

These inequalities help us determine whether an estimate is an underestimate or an overestimate.

19.3. The Distance Problem

Area is useful for much more than finding geometric regions.

Suppose an object moves along a straight line and its velocity is known.

Recall that if velocity is constant, then

$$\text{distance} = \text{velocity} \times \text{time}.$$

For example, if a car travels at

$$60 \text{ mi/hr}$$

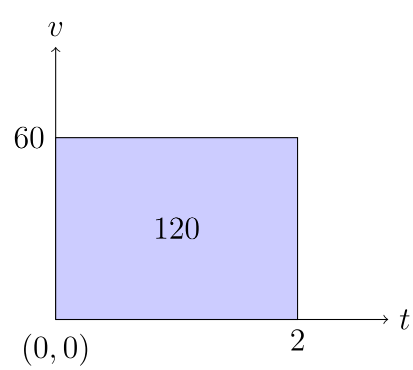
for

$$2 \text{ hours},$$

then the distance traveled is

$$60(2) = 120 \text{ miles}.$$

Geometrically, this can be viewed as the area of a rectangle.



Since

$$\text{Area} = (60)(2) = 120,$$

the distance traveled equals the area under the velocity graph.

Distance from Velocity

If $v(t) \geq 0$ on an interval $[a, b]$, then the distance traveled from time a to time b is equal to the area under the velocity graph.
That is,

$$\text{Distance} = \text{Area under } v(t).$$

Example 19.3.
A car travels with velocity

$$v(t) = 40$$

miles per hour for three hours. Use area to determine the distance traveled.
Solution.
The velocity graph is a horizontal line at $v = 40$.
The distance traveled equals the area of the rectangle:

$$\text{Distance} = (40)(3) = 120.$$

Therefore,

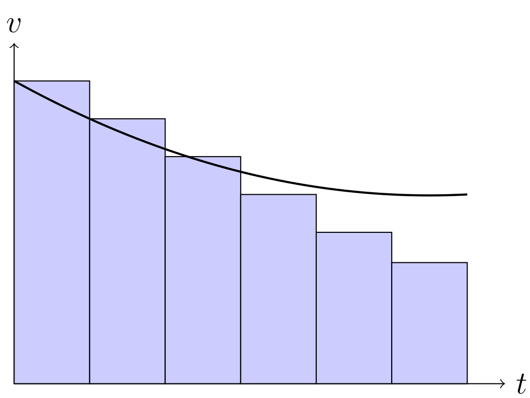
$$\boxed{120 \text{ miles}}.$$

19.3.1 Distance from a Velocity Graph

When velocity is not constant, the distance traveled is no longer the area of a rectangle.

Instead, we approximate the area under the velocity curve using rectangles.

As we use more rectangles, the approximation becomes more accurate.



Distance ≈ Area Under Velocity Curve

Example 19.4.

Estimate the distance traveled during the first 4 seconds if the velocity values are given by

t	0	1	2	3	4
$v(t)$	20	18	15	11	8

using a right-endpoint approximation.

Solution.

There are four subintervals, each having width

$$\Delta t = 1.$$

Using right endpoints,

$$R_4 = (18 + 15 + 11 + 8)(1).$$

Thus,

$$R_4 = 52.$$

Therefore, the estimated distance traveled is

$$\boxed{52}.$$

Warning

The area under a velocity graph represents distance traveled only when

$$v(t) \geq 0.$$

If the velocity becomes negative, then the area below the t-axis must be treated differently.

We will discuss this distinction in a later section.

19.3.2 Summary

Important

1. Area under a curve can be approximated using rectangles.
2. The width of each rectangle is

$$\Delta x = \frac{b-a}{n}.$$

3. A Riemann Sum has the form

$$f(x_1)\Delta x + f(x_2)\Delta x + \cdots + f(x_n)\Delta x.$$

4. Left and right endpoint approximations provide estimates of area.
5. Distance traveled is equal to the area under a velocity graph.
6. More rectangles generally produce more accurate approximations.

Exercises

1. Estimate the area under
$$f(x) = x^2$$
on $[0, 2]$ using four rectangles and left endpoints.

2. Estimate the area under
$$f(x) = x^2$$
on $[0, 2]$ using four rectangles and right endpoints.

3. Estimate the area under
$$f(x) = 4 - x$$
on $[0, 4]$ using four rectangles and left endpoints.

4. Estimate the area under
$$f(x) = 4 - x$$
on $[0, 4]$ using four rectangles and right endpoints.

5. Estimate the area under
$$f(x) = x + 1$$
on $[0, 5]$ using five rectangles and left endpoints.

6. Estimate the area under
$$f(x) = x + 1$$
on $[0, 5]$ using five rectangles and right endpoints.

7. Determine whether the left-endpoint approximation is an overestimate or underestimate for
$$f(x) = x^3 \quad \text{on} \quad [0, 2].$$

8. Determine whether the right-endpoint approximation is an overestimate or underestimate for
$$f(x) = x^3$$
on $[0, 2]$.

9. Determine whether the left-endpoint approximation is an overestimate or underestimate for
$$f(x) = 5 - x.$$

10. Determine whether the right-endpoint approximation is an overestimate or underestimate for
$$f(x) = 5 - x.$$

11. The velocity of a car is given by
$$v(t) = 50$$
miles per hour. Find the distance traveled in 4 hours.

12. The velocity of a train is
$$v(t) = 70$$
miles per hour. Find the distance traveled in 3 hours.

13. Use the following table to estimate the distance traveled using a right-endpoint approximation.

t	0	1	2	3	4
$v(t)$	15	18	20	21	24

14. Use the following table to estimate the distance traveled using a left-endpoint approximation.

t	0	1	2	3	4
$v(t)$	15	18	20	21	24

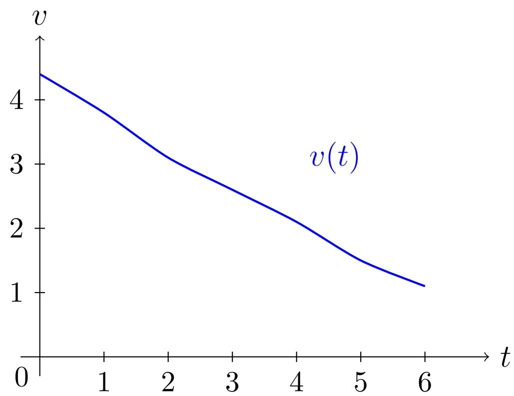
15. Sketch a decreasing function and illustrate a left-endpoint approximation using four rectangles.

16. Sketch a decreasing function and illustrate a right-endpoint approximation using four rectangles.

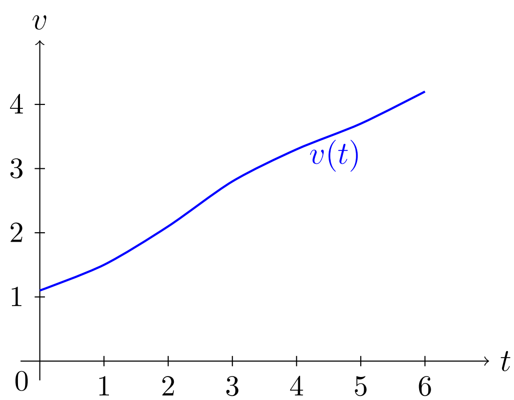
17. Sketch an increasing function and illustrate a left-endpoint approximation using four rectangles.

18. Sketch an increasing function and illustrate a right-endpoint approximation using four rectangles.

19. The graph below represents velocity versus time. Estimate the distance traveled using six rectangles and left endpoints.



20. The graph below represents velocity versus time. Estimate the distance traveled using six rectangles and right endpoints.



Chapter 20

The Definite Integral

20.1. The Definite Integral

Long before calculus was developed, mathematicians attempted to compute areas using increasingly finer geometric approximations. Archimedes (287–212 BC) developed the "method of exhaustion," which approximated areas by polygons with more and more sides. Nearly two thousand years later, Newton and Leibniz generalized this idea into the modern concept of the definite integral.

In Section 4.1 we approximated areas and distances using rectangles. As the number of rectangles increases, these approximations become more accurate. The **definite integral** is defined as the limit of these approximations and provides a powerful tool for measuring area, net area, accumulated change, and many other quantities in mathematics and science.

The definite integral arises from a special type of limit called a **Riemann Sum**.

Definition: Definite Integral

Let f be defined on the interval $[a, b]$.
Divide the interval into n equal sub-intervals of width

$$\Delta x = \frac{b-a}{n}.$$

Let

$$x_0 = a, \qquad x_n = b$$

and choose a sample point x_i^* from each sub-interval.
Provided the limit exists, the **definite integral** of f from a to b is

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x,$$

Important

The symbol
$$\sum_{i=1}^{n} f(x_i^*)\Delta x$$
is called a **Riemann Sum**.

20.1.1 Notation

Integral Notation

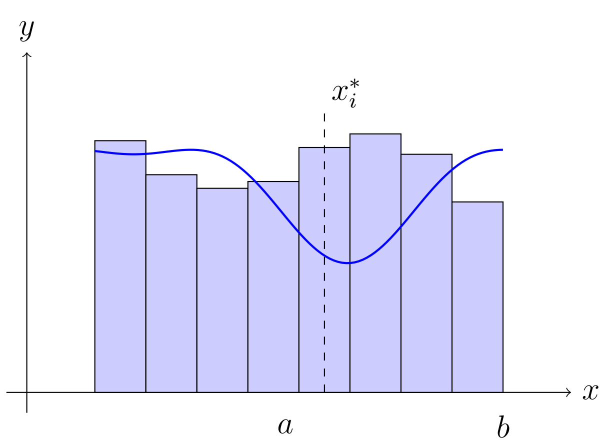
In the expression
$$\int_a^b f(x)\,dx,$$

- $f(x)$ is called the **integrand**,
- a is the **lower limit of integration**,
- b is the **upper limit of integration**,
- dx indicates the variable of integration.

Important

A definite integral is a **number**.
It does not depend on the variable used.
$$\int_a^b f(x)\,dx = \int_a^b f(t)\,dt = \int_a^b f(r)\,dr.$$

The definite integral is obtained by taking the limit of the areas of approximating rectangles.



A Riemann sum approximates the area using rectangles.

20.1.2 The Area Interpretation

Suppose $f(x) \geq 0$ on $[a, b]$.

Then, every rectangle contributes a positive area. As the widths of the rectangles approach zero, the Riemann sum approaches the exact area under the curve.

Area Interpretation

If f is continuous and

$$f(x) \geq 0$$

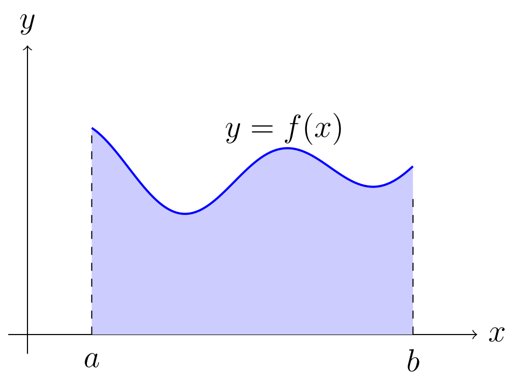
for all x in $[a, b]$, then

$$\int_a^b f(x)\,dx$$

represents the area of the region bounded by

$$y = f(x), \qquad x = a, \qquad x = b,$$

and the x-axis.



20.1.3 Net Area

Not all functions remain above the x-axis. If a graph crosses the x-axis, some portions contribute positive area, and some contribute negative area.

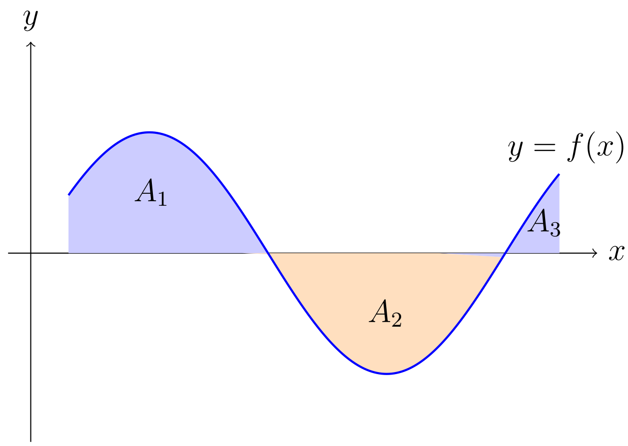
Net Area

For a function that takes both positive and negative values,

$$\int_a^b f(x)\,dx$$

represents the **net area**

$$(\text{area above the } x\text{-axis}) - (\text{area below the } x\text{-axis}).$$



In this situation,

$$\int_a^b f(x)\,dx = A_1 + A_3 - A_2.$$

20.1.4 Functions That Are Integrable

A bounded function f on a closed interval $[a, b]$ is called **Riemann integrable** if the area under its graph can be approximated as closely as we want by using rectangles. Fortunately, nearly every function encountered in Calculus I is integrable.

Integrability Theorem

If a function is continuous on a closed interval $[a, b]$, then it is integrable on $[a, b]$.

Important

For this course, whenever a function is continuous on a closed interval, you may assume its definite integral exists.

20.2. Evaluating Definite Integrals from the Definition

Although the definition of the definite integral involves a limit of Riemann sums, evaluating that limit directly can be algebraically intensive. Nevertheless, understanding the process helps explain where the definite integral comes from. To simplify calculations, we often choose the sample points to be the **right endpoints** of the subintervals.

Formula

For a partition of $[a, b]$ into n equal subintervals,

$$\Delta x = \frac{b-a}{n},$$

and the right endpoints are

$$x_i = a + i\Delta x.$$

Then

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x.$$

20.2.1 Procedure for Evaluating an Integral from the Definition

Important

To evaluate

$$\int_a^b f(x)\,dx$$

from the definition:

1. Compute

$$\Delta x = \frac{b-a}{n}.$$

2. Determine the right endpoints

$$x_i = a + i\Delta x.$$

3. Substitute x_i into $f(x)$.
4. Form the Riemann sum

$$\sum_{i=1}^{n} f(x_i)\Delta x.$$

5. Simplify the sum.
6. Take the limit as $n \to \infty$.

Example 20.1.
Evaluate

$$\int_0^2 x\,dx$$

using the definition of the definite integral.
Solution.
First,

$$\Delta x = \frac{2-0}{n} = \frac{2}{n}.$$

The right endpoints are

$$x_i = \frac{2i}{n}.$$

Therefore

$$f(x_i) = \frac{2i}{n}.$$

The Riemann sum becomes

$$\sum_{i=1}^{n} \frac{2i}{n} \cdot \frac{2}{n} = \frac{4}{n^2} \sum_{i=1}^{n} i.$$

Using

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2},$$

we obtain

$$\frac{4}{n^2} \cdot \frac{n(n+1)}{2} = 2\left(\frac{n+1}{n}\right).$$

Taking the limit,

$$\int_0^2 x\,dx = \lim_{n\to\infty} 2\left(\frac{n+1}{n}\right) = 2.$$

$$\boxed{\int_0^2 x\,dx = 2}$$

Example 20.2.
Evaluate

$$\int_0^1 x^2\,dx$$

using the definition.
Solution.
We have

$$\Delta x = \frac{1}{n}, \qquad x_i = \frac{i}{n}.$$

Thus

$$f(x_i) = \left(\frac{i}{n}\right)^2.$$

The Riemann sum is

$$\sum_{i=1}^{n} \left(\frac{i}{n}\right)^2 \frac{1}{n} = \frac{1}{n^3}\sum_{i=1}^{n} i^2.$$

Recall

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}.$$

Therefore

$$\frac{1}{n^3}\cdot\frac{n(n+1)(2n+1)}{6}.$$

Taking the limit,

$$\int_0^1 x^2\,dx = \frac{1}{6}\lim_{n\to\infty}\frac{(n+1)(2n+1)}{n^2}.$$

$$= \frac{1}{6}(2) = \frac{1}{3}.$$

$$\boxed{\int_0^1 x^2\,dx = \frac{1}{3}}$$

20.3. The Midpoint Rule

Computing limits of Riemann sums exactly is often difficult.

Instead, we frequently use Riemann sums to **approximate** definite integrals.

One of the most accurate elementary methods is the **Midpoint Rule**.

Formula

The Midpoint Rule states that

$$\int_a^b f(x)\,dx \approx \sum_{i=1}^{n} f(\overline{x}_i)\Delta x,$$

where

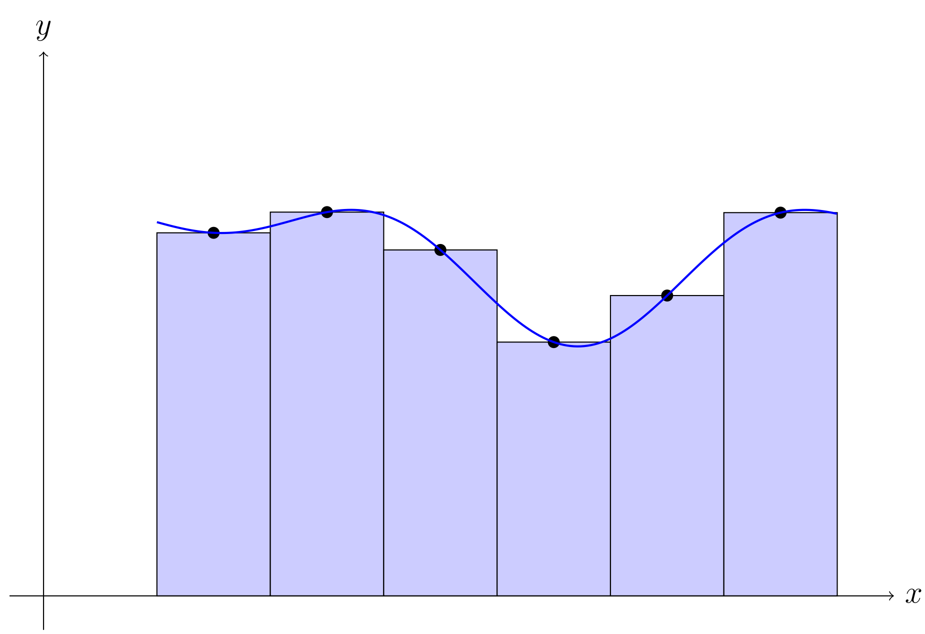
$$\Delta x = \frac{b-a}{n}$$

and

$$\overline{x}_i = \frac{x_{i-1}+x_i}{2}$$

is the midpoint of the ith subinterval.

20.3.1 A Picture of the Midpoint Rule



Rectangle heights determined by midpoint values

Example 20.3.
Use the Midpoint Rule with $n = 4$ to approximate

$$\int_0^4 x^2\,dx.$$

Solution.
First,

$$\Delta x = \frac{4-0}{4} = 1.$$

The subintervals are

$$[0,1], [1,2], [2,3], [3,4].$$

Their midpoints are

$$0.5, \quad 1.5, \quad 2.5, \quad 3.5.$$

Therefore

$$M_4 = 1\Big[(0.5)^2 + (1.5)^2 + (2.5)^2 + (3.5)^2\Big].$$

$$= 0.25 + 2.25 + 6.25 + 12.25 = 21.$$

Thus

$$\boxed{\int_0^4 x^2\,dx \approx 21}$$

using the Midpoint Rule.
(The exact value is $\dfrac{64}{3} \approx 21.33$.)

Warning

The Midpoint Rule produces an approximation, not an exact answer.
Increasing the number of rectangles generally improves the accuracy.

20.4. Properties of the Definite Integral

The following properties allow us to simplify and evaluate definite integrals without returning to the definition every time.

Properties of the Definite Integral

Let f and g be integrable on $[a, b]$, and let c be a constant.

1. $$\int_a^b c\,dx = c(b-a)$$

2. $$\int_a^b [f(x) + g(x)]\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$$

3. $$\int_a^b [f(x) - g(x)]\,dx = \int_a^b f(x)\,dx - \int_a^b g(x)\,dx$$

4. $$\int_a^b cf(x)\,dx = c\int_a^b f(x)\,dx$$

5. $$\int_a^a f(x)\,dx = 0$$

6. $$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$$

7. If

   $$a < c < b,$$

   then

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx.$$

20.4.1 Comparison Properties

The next properties help us estimate the value of an integral.

Comparison Properties

Suppose f and g are integrable on $[a, b]$.

1. If
$$f(x) \geq 0$$
on $[a, b]$, then
$$\int_a^b f(x)\,dx \geq 0.$$
2. If
$$f(x) \geq g(x)$$
on $[a, b]$, then
$$\int_a^b f(x)\,dx \geq \int_a^b g(x)\,dx.$$
3. If
$$m \leq f(x) \leq M$$
for all x in $[a, b]$, then
$$m(b-a) \leq \int_a^b f(x)\,dx \leq M(b-a).$$

20.5. Geometry and Definite Integrals

Sometimes, a definite integral can be evaluated without any calculus. If the graph is made from familiar geometric shapes, we can interpret the integral as the area.

Important

For a function above the x-axis,
$$\int_a^b f(x)\,dx = \text{Area}.$$
For a function below the x-axis,
$$\int_a^b f(x)\,dx = -\text{Area}.$$

Example 20.4.

Evaluate

$$\int_0^4 3\,dx.$$

Solution.

The graph is a rectangle with

$$\text{height} = 3, \qquad \text{width} = 4.$$

Therefore

$$\int_0^4 3\,dx = 3(4) = 12.$$

$$\boxed{12}$$

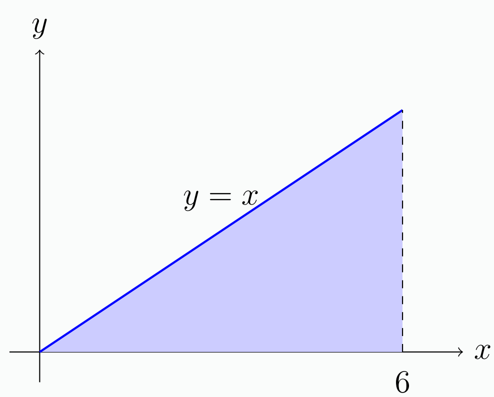
Example 20.5.

Evaluate

$$\int_0^6 x\,dx.$$

Solution.

The graph of $y = x$ from $x = 0$ to $x = 6$ forms a right triangle.



$$\text{base} = 6, \qquad \text{height} = 6.$$

Thus,

$$\int_0^6 x\,dx = \frac{1}{2}(6)(6) = 18.$$

$$\boxed{18}$$

Example 20.6.

Evaluate

$$\int_{-2}^{2} \sqrt{4-x^2}\,dx.$$

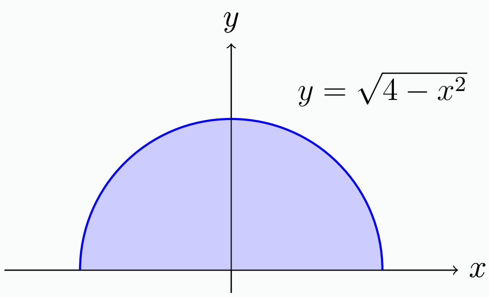
Solution.

The graph

$$y = \sqrt{4-x^2}$$

is the upper semicircle of radius

$$r = 2.$$



Therefore,

$$\int_{-2}^{2} \sqrt{4-x^2}\,dx = \frac{1}{2}\pi r^2 = \frac{1}{2}\pi(2)^2.$$

Hence,

$$\boxed{2\pi}.$$

Example 20.7.

Use properties of integrals to evaluate

$$\int_{1}^{4} (2x^2 - 3x + 1)\,dx$$

given that

$$\int_{1}^{4} x^2\,dx = 21, \qquad \int_{1}^{4} x\,dx = \frac{15}{2}, \qquad \int_{1}^{4} 1\,dx = 3.$$

Solution.

Using linearity,

$$\int_{1}^{4} (2x^2 - 3x + 1)\,dx = 2\int_{1}^{4} x^2\,dx - 3\int_{1}^{4} x\,dx + \int_{1}^{4} 1\,dx.$$

Substituting,

$$= 2(21) - 3\left(\frac{15}{2}\right) + 3.$$

$$= 42 - \frac{45}{2} + 3.$$

$$= \frac{84 - 45 + 6}{2} = \frac{45}{2}.$$

$$\boxed{\frac{45}{2}}$$

20.5.1 Estimating an Integral Using Bounds

Example 20.8.

Suppose

$$2 \le f(x) \le 5$$

for all

$$1 \le x \le 4.$$

Find bounds for

$$\int_1^4 f(x)\,dx.$$

Solution.

Using Property 3,

$$m(b-a) \le \int_1^4 f(x)\,dx \le M(b-a).$$

Substituting

$$m = 2, \quad M = 5, \quad b - a = 3,$$

gives

$$2(3) \le \int_1^4 f(x)\,dx \le 5(3).$$

Thus,

$$\boxed{6 \le \int_1^4 f(x)\,dx \le 15}$$

Warning

A definite integral represents **net area**, not necessarily total area.

If part of the graph lies below the x-axis, those regions contribute negatively to the integral.

20.6. Summary

Important

Key Ideas from Section 4.2

- The definite integral is defined as a limit of Riemann sums:

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x.$$

- If $f(x) \geq 0$ on $[a,b]$, then

$$\int_a^b f(x)\,dx$$

represents the area under the curve.

- If $f(x)$ changes sign, then

$$\int_a^b f(x)\,dx$$

represents net area.

- A definite integral is a number.
- The value of a definite integral does not depend on the variable used.
- Definite integrals satisfy linearity properties.
- Geometric formulas can often be used to evaluate definite integrals.
- The Midpoint Rule provides an approximation to a definite integral.

Exercises

A. Evaluating Definite Integrals from the Definition

Evaluate the following using the definition of the definite integral.

1. $$\int_0^3 x\,dx$$

2. $$\int_0^1 x^2\,dx$$

3. $$\int_1^3 (2x+1)\,dx$$

4. $$\int_0^2 (x^2+1)\,dx$$

B. Midpoint Rule

Use the Midpoint Rule to approximate each integral.

5. $$\int_0^4 x^2\,dx, \qquad n=4$$

6. $$\int_1^5 x^3\,dx, \qquad n=4$$

7. $$\int_0^6 (x+2)\,dx, \qquad n=6$$

8. $$\int_0^\pi \sin x\,dx, \qquad n=4$$

9. $$\int_0^2 e^x\,dx, \qquad n=4$$

C. Using Properties of Integrals

Evaluate.

10.
$$\int_0^5 (3x-2)\,dx$$

given

$$\int_0^5 x\,dx = \frac{25}{2}.$$

11.
$$\int_1^4 (2x^2+3x)\,dx$$

given

$$\int_1^4 x^2\,dx = 21, \qquad \int_1^4 x\,dx = \frac{15}{2}.$$

12.
$$\int_2^6 \big(4f(x) - g(x)\big)\,dx$$

given

$$\int_2^6 f(x)\,dx = 8, \qquad \int_2^6 g(x)\,dx = 5.$$

13.
$$\int_3^3 f(x)\,dx$$

14.
$$\int_5^1 x^2\,dx$$

given

$$\int_1^5 x^2\,dx = 124/3.$$

D. Geometric Evaluation

Evaluate each integral geometrically.

15.
$$\int_0^8 5\,dx$$

16.
$$\int_0^6 x\,dx$$

17.
$$\int_{-3}^3 \sqrt{9-x^2}\,dx$$

18.
$$\int_{-4}^4 \sqrt{16-x^2}\,dx$$

19.
$$\int_0^2 (4-2x)\,dx$$

20.
$$\int_{-2}^2 (2-|x|)\,dx$$

E. Net Area

For each graph, determine whether the definite integral is positive, negative, or zero. Explain.

21. A graph lying entirely above the x-axis.
22. A graph lying entirely below the x-axis.
23. A graph with equal areas above and below the x-axis.
24. A graph with more area below the x-axis than above.

F. Comparison Properties

Find bounds for the following integrals.

25.
$$\int_2^7 f(x)\,dx$$
if
$$1 \le f(x) \le 4.$$

26.
$$\int_0^3 f(x)\,dx$$
if
$$2 \le f(x) \le 5.$$

27.
$$\int_1^5 f(x)\,dx$$
if
$$-1 \le f(x) \le 3.$$

G. True or False

Determine whether each statement is true or false.

28.
$$\int_a^a f(x)\,dx = 0.$$

29.
$$\int_a^b f(x)\,dx = \int_b^a f(x)\,dx.$$

30.
$$\int_a^b [f(x) + g(x)]\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx.$$

31. If
$$f(x) \ge 0$$

on $[a, b]$, then

$$\int_a^b f(x)\,dx \geq 0.$$

32. A definite integral is always positive.

33. $$\int_a^b c\,dx = c(b-a).$$

H. Conceptual Questions

34. Explain the difference between area and net area.

35. Why is a definite integral a number while an indefinite integral is a function?

36. What role does

$$\Delta x$$

play in a Riemann sum?

37. Why does increasing the number of rectangles improve the accuracy of a Riemann sum?

38. Explain why

$$\int_a^b f(x)\,dx = \int_a^b f(t)\,dt.$$

39. Describe the geometric meaning of

$$\int_a^b f(x)\,dx.$$

Chapter 21

The Fundamental Theorem of Calculus

21.1. The Fundamental Theorem of Calculus (TFTC)

Why Is This Theorem So Important?

Up to this point, differentiation and integration have appeared to be completely different processes.
The Fundamental Theorem of Calculus reveals that these two operations are actually inverses of one another.
Many mathematicians consider this theorem to be one of the most beautiful results in all of mathematics because it unifies two seemingly unrelated ideas.

In Chapter 20, we learned that a definite integral can be interpreted as an area or a net area. In this section, we discover the remarkable relationship between integration and differentiation.

The Fundamental Theorem of Calculus (TFTC) is one of the most important results in mathematics. It establishes that differentiation and integration are inverse processes.

The theorem has two parts:

- TFTC Part 1 explains how to differentiate functions defined by definite integrals.
- TFTC Part 2 provides an efficient way to evaluate definite integrals using antiderivatives.

Together, these results form the foundation of modern calculus.

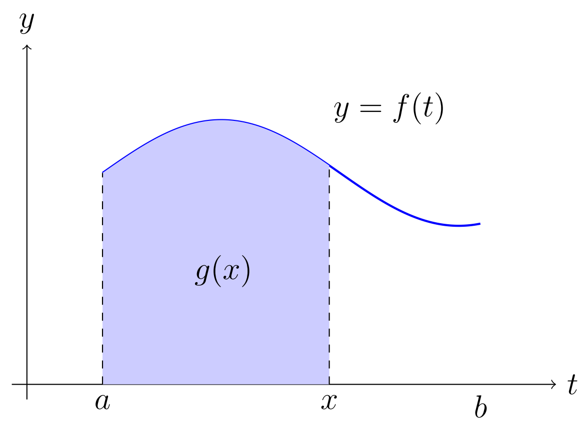
21.1.1 Functions Defined by Integrals

Suppose f is continuous on an interval $[a, b]$. We can define a new function

$$g(x) = \int_a^x f(t)\,dt, \qquad a \le x \le b.$$

Notice that the variable of integration is t, while the variable defining the function is x.

If $f(t) \ge 0$, then $g(x)$ represents the area under the graph of f from $t = a$ to $t = x$. For this reason, $g(x)$ is sometimes called an **accumulation function** or an **area-so-far function**.



The value of $g(x)$ is the area accumulated from a to x.

21.1.2 The Fundamental Theorem of Calculus (Part 1)

Theorem 21.1 (The Fundamental Theorem of Calculus: Part 1)

Suppose f is continuous on $[a, b]$ and define

$$g(x) = \int_a^x f(t)\,dt.$$

Then g is differentiable on (a, b) and

$$g'(x) = f(x).$$

In other words,

$$\frac{d}{dx}\left(\int_a^x f(t)\,dt\right) = f(x).$$

Important

The derivative of a definite integral with respect to its upper limit equals the integrand evaluated at the upper limit.

Why FTC Part 1 Makes Sense

Suppose

$$g(x) = \int_a^x f(t)\,dt.$$

If we increase x by a very small amount Δx, then the additional area added is approximately a rectangle with

$$\text{height} = f(x) \qquad \text{and} \qquad \text{width} = \Delta x.$$

Therefore,

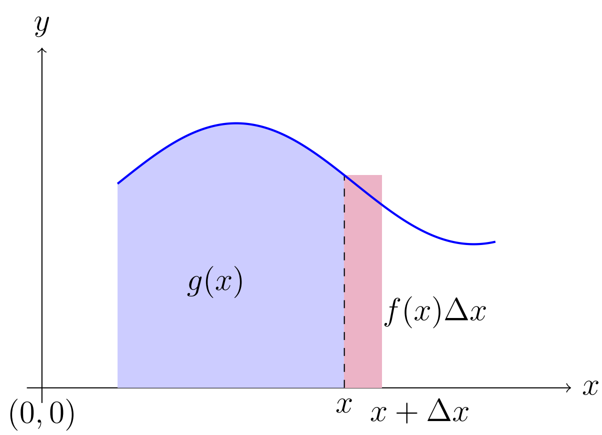
$$g(x + \Delta x) - g(x) \approx f(x)\Delta x.$$

Dividing by Δx gives

$$\frac{g(x + \Delta x) - g(x)}{\Delta x} \approx f(x).$$

Taking the limit as $\Delta x \to 0$ yields

$$g'(x) = f(x).$$



Example 21.1.

Find

$$\frac{d}{dx}\left[\int_0^x (t^3 + 2t)\,dt\right].$$

Solution.

$$\frac{d}{dx}\left[\int_0^x (t^3+2t)\,dt\right] = x^3+2x.$$

$$\boxed{x^3+2x.}$$

Example 21.2.
Find

$$\frac{d}{dx}\left(\int_1^x \sqrt{1+t^2}\,dt\right).$$

Solution.

$$\frac{d}{dx}\left(\int_1^x \sqrt{1+t^2}\,dt\right) = \sqrt{1+x^2}.$$

$$\boxed{\sqrt{1+x^2}}.$$

Example 21.3.
Let

$$g(x) = \int_{-2}^x \frac{1}{1+t^2}\,dt.$$

Find $g'(x)$.
Solution.

$$g'(x) = \frac{d}{dx}\left[\int_{-2}^x \frac{1}{1+t^2}\,dt\right] = \frac{1}{1+x^2}.$$

$$\boxed{\frac{1}{1+x^2}.}$$

21.1.3 The Fundamental Theorem of Calculus (Part 2)

TFTC Part 1 tells us how to differentiate functions defined by integrals. The second part of the theorem tells us how to evaluate definite integrals efficiently. Before the Fundamental Theorem of Calculus was discovered, finding exact areas was often very difficult. TFTC Part 2 reduces the problem of evaluating a definite integral to finding an antiderivative.

Theorem 21.2 (The Fundamental Theorem of Calculus: Part 2)

Let f be continuous on $[a, b]$ and let F be any antiderivative of f, that is,

$$F'(x) = f(x).$$

Then,

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

Important

To evaluate a definite integral:

1. Find an antiderivative $F(x)$ of the integrand.
2. Evaluate $F(b) - F(a)$.

21.1.4 Evaluation Notation

A convenient notation for evaluating definite integrals is

$$\int_a^b f(x)\,dx = [F(x)]_a^b\,.$$

This means

$$[F(x)]_a^b = F(b) - F(a).$$

For example,

$$\left[\frac{x^3}{3}\right]_1^4 = \frac{4^3}{3} - \frac{1^3}{3}.$$

Example: Evaluating a Polynomial Integral

Example 21.4.
Evaluate

$$\int_1^3 (x^2 + 2x)\,dx.$$

Solution.

$$\begin{aligned}\int_1^3 (x^2+2x)\,dx &= \left[\frac{x^3}{3}+x^2\right]_1^3 \\ &= \left(\frac{3^3}{3}+3^2\right)-\left(\frac{1^3}{3}+1^2\right) \\ &= 18-\frac{4}{3} \\ &= \frac{50}{3}.\end{aligned}$$

$$\boxed{\int_1^3 (x^2+2x)\,dx = \frac{50}{3}.}$$

Example: A Trigonometric Integral

Example 21.5.
Evaluate

$$\int_0^{\pi/2} \cos x\,dx.$$

Solution.
Using the part two of the Fundamental Theorem of Calculus,

$$\int_0^{\pi/2} \cos x\,dx = [\sin x]_0^{\pi/2}.$$

$$= \sin\left(\frac{\pi}{2}\right) - \sin(0) = 1 - 0 = 1.$$

Therefore,

$$\boxed{\int_0^{\pi/2} \cos x\,dx = 1.}$$

Example: A Rational Function

Example 21.6.
Evaluate

$$\int_1^2 \frac{1}{x^2}\,dx.$$

Solution.

$$\begin{aligned}\int_1^2 \frac{1}{x^2}\,dx &= \left[-\frac{1}{x}\right]_1^2\\ &= \left(-\frac{1}{2}\right)-\left(-\frac{1}{1}\right)\\ &= -\frac{1}{2}+1\\ &= \frac{1}{2}.\end{aligned}$$

Thus,

$$\boxed{\int_1^2 \frac{1}{x^2}\,dx = \frac{1}{2}}.$$

21.2. Net Change Theorem

TFTC Part 2 has an important interpretation in applications. Suppose a quantity Q changes over time and its rate of change is $Q'(t)$. Since Q' is the derivative of Q, TFTC Part 2 gives

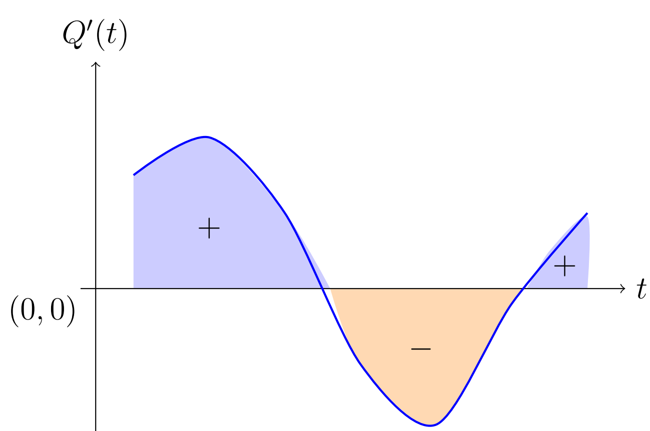
$$\int_a^b Q'(t)\,dt = Q(b) - Q(a).$$

This result is called the **Net Change Theorem**.

Theorem 21.3 (Net Change Theorem)

The integral of a rate of change over an interval equals the net change of the quantity.

$$\int_a^b Q'(t)\,dt = Q(b) - Q(a).$$



Net Change = Positive Area − Negative Area

21.2.1 Interpreting the Net Change Theorem

If $v(t)$ represents velocity, then

$$\int_a^b v(t)\,dt$$

represents the net change in position (displacement).

If $r(t)$ represents the rate at which water flows into a tank, then

$$\int_a^b r(t)\,dt$$

represents the total change in the amount of water in the tank.

More generally:

$$\text{Accumulated Change} = \int (\text{rate of change})\,dt.$$

Example: Position from Velocity

Example 21.7.

A particle moves along a line with velocity

$$v(t) = 3t^2.$$

Find the change in position from $t = 1$ to $t = 3$.

Solution.

The change in position is the definite integral of the velocity:

$$\Delta s = \int_1^3 v(t)\,dt = \int_1^3 3t^2\,dt.$$

Evaluate the integral:

$$= \left[t^3\right]_1^3 = 3^3 - 1^3 = 27 - 1 = 26.$$

Therefore, the change in position is

$$\boxed{26.}$$

Example: Water Flow

Example 21.8.
Water flows into a tank at a rate of

$$r(t) = 4 + t$$

gallons per minute. Find the total amount of water added during the first 3 minutes.
Solution.
The total amount of water added is the definite integral of the flow rate:

$$\int_0^3 r(t)\,dt = \int_0^3 (4 + t)\,dt.$$

Evaluate the integral:

$$\int_0^3 (4 + t)\,dt = \left[4t + \frac{t^2}{2}\right]_0^3 = \left(4(3) + \frac{3^2}{2}\right) - \left(4(0) + \frac{0^2}{2}\right) = 12 + \frac{9}{2} = \frac{33}{2}.$$

Therefore, the total amount of water added during the first 3 minutes is

$$\boxed{\frac{33}{2} \text{ gallons.}}$$

21.3. Differentiation and Integration as Inverse Processes

TFTC Part 1 tells us that differentiation undoes integration, while TFTC Part 2 tells us that integration undoes differentiation. Together, they show that differentiation and integration are inverse processes.

Theorem 21.4 (The Fundamental Theorem of Calculus)

Suppose f is continuous on $[a, b]$.

1. If

$$g(x) = \int_a^x f(t)\,dt,$$

then

$$g'(x) = f(x).$$

2. If F is any antiderivative of f, then

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

Important

Differentiation and integration are inverse operations:

$$\frac{d}{dx}\left(\int_a^x f(t)\,dt\right) = f(x)$$

and

$$\int_a^b F'(x)\,dx = F(b) - F(a).$$

21.3.1 When the Upper Limit is a Function

TFTC Part 1 can be combined with the Chain Rule.

Suppose

$$G(x) = \int_a^{u(x)} f(t)\,dt.$$

Then

$$G'(x) = f(u(x))u'(x).$$

Theorem 21.5 (FTC Part 1 + Chain Rule)

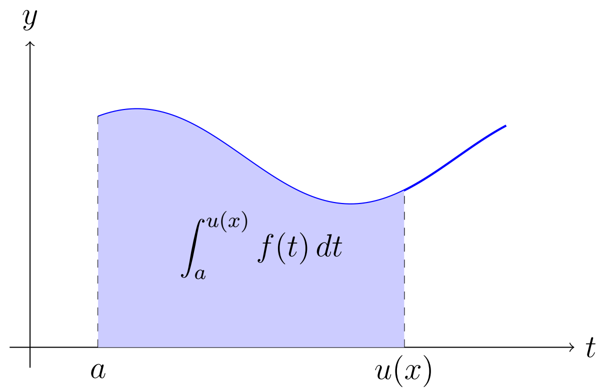
If

$$G(x) = \int_a^{u(x)} f(t)\,dt,$$

then

$$G'(x) = f(u(x))u'(x).$$

21.3.2 Visual Interpretation



As $u(x)$ changes, the accumulated area changes. The rate at which the area changes is the height of the graph at $u(x)$ multiplied by the rate at which the boundary $u(x)$ moves.

Example: Variable Upper Limit

Example 21.9.
Find

$$\frac{d}{dx}\left(\int_0^{x^2} \sqrt{1+t^3}\,dt\right).$$

Solution.

$$\begin{aligned}\frac{d}{dx}\left(\int_0^{x^2} \sqrt{1+t^3}\,dt\right) &= \sqrt{1+x^2}\cdot\frac{d}{dx}x^2\\ &= 2x\sqrt{1+x^2}.\end{aligned}$$

$$\boxed{2x\sqrt{1+x^6}}$$

Example: Trigonometric Upper Limit

Example 21.10.
Find

$$\frac{d}{dx}\left(\int_1^{\sin x} \frac{\sqrt{t}}{1+t}\,dt\right).$$

Solution.

$$\frac{d}{dx}\left(\int_1^{\sin x} \frac{\sqrt{t}}{1+t}\,dt\right) = \frac{\sqrt{\sin x}}{1+\sin x} \cdot \frac{d}{dx}\sin x$$
$$= \frac{\sqrt{\sin x}}{1+\sin x}\cos x.$$

$$\boxed{\frac{\cos x\sqrt{\sin x}}{1+\sin x}}$$

21.3.3 When the Lower Limit is a Function

Suppose
$$H(x) = \int_{v(x)}^{a} f(t)\,dt.$$
Using the fact that
$$\int_{v(x)}^{a} f(t)\,dt = -\int_{a}^{v(x)} f(t)\,dt,$$
we obtain
$$H'(x) = -f(v(x))v'(x).$$

Theorem 21.6 (Variable Lower Limit)

If
$$H(x) = \int_{v(x)}^{a} f(t)\,dt,$$
then
$$H'(x) = -f(v(x))v'(x).$$

Example: Variable Lower Limit

Example 21.11.
Find

$$\frac{d}{dx}\left(\int_x^3 \sqrt{1+t^2}\,dt\right).$$

Solution.
Using the Fundamental Theorem of Calculus Part 1,

$$\frac{d}{dx}\left(\int_x^3 \sqrt{1+t^2}\,dt\right) = -\sqrt{1+x^2}.$$

The negative sign appears because the variable is the *lower* limit of integration.
Therefore,

$$\boxed{\frac{d}{dx}\left(\int_x^3 \sqrt{1+t^2}\,dt\right) = -\sqrt{1+x^2}.}$$

21.3.4 Both Limits are Functions

Suppose

$$F(x) = \int_{v(x)}^{u(x)} f(t)\,dt.$$

Then

$$F(x) = \int_a^{u(x)} f(t)\,dt - \int_a^{v(x)} f(t)\,dt.$$

Differentiating gives

$$F'(x) = f(u(x))u'(x) - f(v(x))v'(x).$$

Theorem 21.7 (Leibniz Rule)

If

$$F(x) = \int_{v(x)}^{u(x)} f(t)\,dt,$$

then

$$F'(x) = f(u(x))u'(x) - f(v(x))v'(x).$$

Example: Both Limits Variable

Example 21.12.
Find

$$\frac{d}{dx}\left(\int_{\sqrt{x}}^{x^3}\cos(t^2)\,dt\right).$$

Solution.
Using the Fundamental Theorem of Calculus together with the Chain Rule,

$$\frac{d}{dx}\left(\int_{\sqrt{x}}^{x^3}\cos(t^2)\,dt\right) = \cos\big((x^3)^2\big)\cdot\frac{d}{dx}(x^3) - \cos\big((\sqrt{x})^2\big)\cdot\frac{d}{dx}(\sqrt{x}).$$

Since

$$(x^3)^2 = x^6, \qquad (\sqrt{x})^2 = x,$$

and

$$\frac{d}{dx}(x^3) = 3x^2, \qquad \frac{d}{dx}(\sqrt{x}) = \frac{1}{2\sqrt{x}},$$

we obtain

$$= 3x^2\cos(x^6) - \frac{\cos x}{2\sqrt{x}}.$$

Therefore,

$$\boxed{\frac{d}{dx}\left(\int_{\sqrt{x}}^{x^3}\cos(t^2)\,dt\right) = 3x^2\cos(x^6) - \frac{\cos x}{2\sqrt{x}}.}$$

21.3.5 Accumulation Functions

Functions defined by integrals occur frequently in applications.

A function of the form

$$A(x) = \int_a^x f(t)\,dt$$

is called an **accumulation function** because it accumulates area (or accumulated change) from a to x.

By TFTC Part 1,

$$A'(x) = f(x).$$

Thus:

- $A(x)$ measures the accumulated quantity.
- $A'(x)$ measures the rate at which that quantity accumulates.

Example: Tangent Line to an Accumulation Function

Example 21.13.
Let

$$F(x) = \int_{\pi}^{x} \frac{\cos t}{t}\, dt.$$

Find the equation of the tangent line at $x = \pi$.
Solution.
To find the equation of the tangent line, we need the point and the slope.
First, find the point:

$$F(\pi) = \int_{\pi}^{\pi} \frac{\cos t}{t}\, dt = 0.$$

Next, use the Fundamental Theorem of Calculus Part 1 to find the derivative:

$$F'(x) = \frac{\cos x}{x}.$$

Thus,

$$F'(\pi) = \frac{\cos \pi}{\pi} = -\frac{1}{\pi}.$$

Using the point-slope form with the point $(\pi, 0)$,

$$y - 0 = -\frac{1}{\pi}(x - \pi).$$

Therefore, the equation of the tangent line is

$$\boxed{y = -\frac{1}{\pi}(x - \pi).}$$

21.3.6 Increasing and Decreasing Accumulation Functions

If

$$F(x) = \int_a^x f(t)\,dt,$$

then

$$F'(x) = f(x).$$

Therefore:

- If $f(x) > 0$, then F is increasing.
- If $f(x) < 0$, then F is decreasing.
- Critical points of F occur where $f(x) = 0$.

Example: Determining Intervals of Increase

Example 21.14.
Let

$$F(x) = \int_0^x (1 - t^2)\cos^2 t\,dt.$$

Determine where F is increasing.
Solution.
By the Fundamental Theorem of Calculus,

$$F'(x) = (1 - x^2)\cos^2 x.$$

The function F is increasing where

$$F'(x) > 0.$$

Since

$$\cos^2 x \geq 0,$$

the sign of $F'(x)$ is determined by $1 - x^2$. We have

$$1 - x^2 > 0$$

when

$$-1 < x < 1.$$

Although $\cos^2 x = 0$ at $x = \frac{\pi}{2} + k\pi$, none of these points lie in $(-1, 1)$.
Therefore, F is increasing on

$$\boxed{(-1, 1)}.$$

21.4. Applications of the Fundamental Theorem of Calculus

The Fundamental Theorem of Calculus is one of the most important results in mathematics because it connects derivatives and integrals. In applications, it allows us to recover accumulated quantities from their rates of change.

Recall the Net Change Theorem:

$$\int_a^b F'(x)\,dx = F(b) - F(a).$$

This theorem says that the integral of a rate of change equals the total change in the quantity.

Theorem 21.8 (Net Change Theorem)

If Q is differentiable on $[a, b]$, then

$$\int_a^b Q'(t)\,dt = Q(b) - Q(a).$$

That is,

$$\text{Net Change} = \text{Final Value} - \text{Initial Value}.$$

21.4.1 Velocity and Position

Suppose a particle moves along a line and its position is $s(t)$.

Then

$$s'(t) = v(t),$$

where $v(t)$ is the velocity.

Applying the Net Change Theorem gives

$$\int_a^b v(t)\,dt = s(b) - s(a).$$

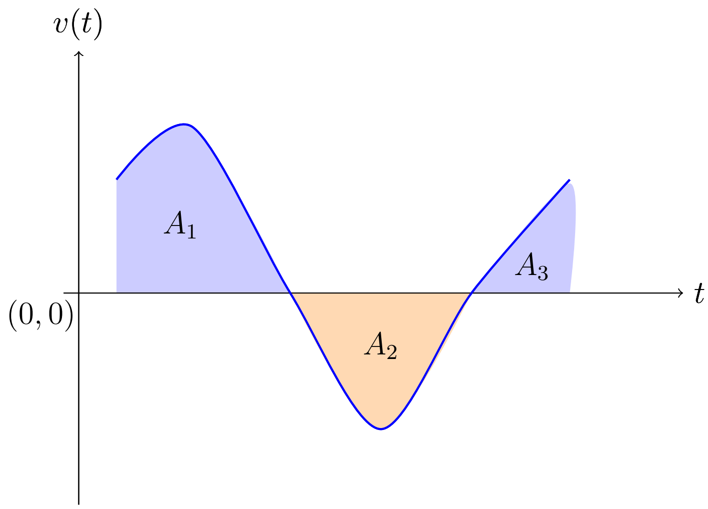
Thus:

- $\displaystyle\int_a^b v(t)\,dt$ gives the **displacement**.
- Displacement may be positive, negative, or zero.

Important

Displacement is not necessarily the same as total distance traveled.
When velocity changes sign, positive and negative contributions cancel.

21.4.2 Velocity Graph and Displacement



For this graph,

$$\int_a^b v(t)\,dt = A_1 - A_2 + A_3.$$

The negative area contributes negatively because the velocity is negative.

Example: Change in Position

Example 21.15.
A particle has velocity

$$v(t) = 3t^2 - 6t.$$

Find the change in position from $t = 0$ to $t = 3$.
Solution.
The change in position is given by the Net Change Theorem:

$$\Delta s = \int_0^3 v(t)\,dt = \int_0^3 (3t^2 - 6t)\,dt.$$

An antiderivative of $3t^2 - 6t$ is

$$t^3 - 3t^2.$$

Therefore,

$$\Delta s = \left[t^3 - 3t^2\right]_0^3.$$

Evaluating,

$$\begin{aligned} &= \left(3^3 - 3(3^2)\right) - \left(0^3 - 3(0^2)\right) \\ &= (27 - 27) - 0 \\ &= 0. \end{aligned}$$

Thus, the change in position is

$$\boxed{0}.$$

Accumulation Functions and Their Graphs

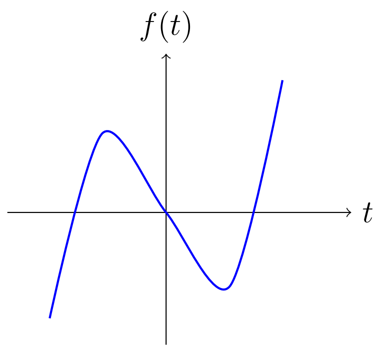
Suppose

$$F(x) = \int_a^x f(t)\,dt.$$

Then, TFTC Part 1 gives

$$F'(x) = f(x).$$

This fact allows us to determine where F is increasing, decreasing, concave up, and concave down directly from the graph of f.



Since

$$F'(x) = f(x),$$

we obtain:

- F increases where $f > 0$.
- F decreases where $f < 0$.
- Critical points of F occur where $f = 0$.

Furthermore,

$$F''(x) = f'(x).$$

Thus:

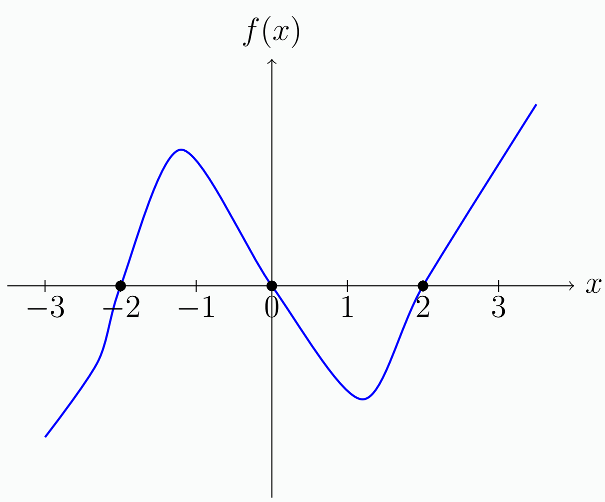
- F is concave up where f is increasing.
- F is concave down where f is decreasing.

Example: Interpreting a Graph

Example 21.16.
Let

$$F(x) = \int_1^x f(t)\,dt,$$

where f is the function shown below.



Determine intervals where F is increasing and intervals where F is concave downward.

Solution.
By the Fundamental Theorem of Calculus,

$$F'(x) = f(x).$$

Therefore, F is increasing where

$$F'(x) > 0.$$

Since

$$F'(x) = f(x),$$

F is increasing where

$$f(x) > 0.$$

From the graph, this occurs on

$$\boxed{(-2, 0) \cup (2, \infty)}.$$

Next,

$$F''(x) = f'(x).$$

Therefore, F is concave downward where

$$F''(x) < 0.$$

Since

$$F''(x) = f'(x),$$

F is concave downward where f is decreasing.
From the graph, f is decreasing on

$$\boxed{(-2, 2)}.$$

Therefore,

$$\boxed{F \text{ is increasing on } (-2, 0) \cup (2, \infty)}$$

and

$$\boxed{F \text{ is concave downward on } (-2, 2).}$$

21.4.3 Riemann Sums and Definite Integrals

One of the most important applications of the FTC is evaluating limits that arise from Riemann sums.

Recall that

$$\int_a^b f(x)\,dx = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i^*)\Delta x.$$

To evaluate such limits:

1. Identify Δx.
2. Identify the interval $[a, b]$.
3. Identify the function $f(x)$.
4. Rewrite the limit as a definite integral.
5. Evaluate using TFTC Part 2.

Example: A Square-Root Riemann Sum

Example 21.17.
Evaluate

$$\lim_{n\to\infty}\frac{1}{n}\left(\sqrt{\frac{1}{n}}+\sqrt{\frac{2}{n}}+\cdots+\sqrt{\frac{n}{n}}\right).$$

Solution.
Rewrite the expression using summation notation:

$$\lim_{n\to\infty}\sum_{i=1}^{n}\sqrt{\frac{i}{n}}\cdot\frac{1}{n}.$$

This is a Riemann sum with

$$\Delta x = \frac{1}{n}$$

on the interval

$$[0, 1].$$

Also,

$$x_i = \frac{i}{n}.$$

So the limit becomes

$$\int_0^1 \sqrt{x}\, dx.$$

Now evaluate:

$$\int_0^1 \sqrt{x}\, dx = \int_0^1 x^{1/2}\, dx.$$

$$= \left[\frac{x^{3/2}}{3/2}\right]_0^1 = \left[\frac{2}{3}x^{3/2}\right]_0^1 = \frac{2}{3}(1)^{3/2} - \frac{2}{3}(0)^{3/2} = \frac{2}{3}.$$

Therefore,

$$\boxed{\frac{2}{3}}.$$

21.5. Summary of Important Facts

Important

If

$$F(x) = \int_a^x f(t)\, dt,$$

then

$$F'(x) = f(x).$$

If $F'(x) = f(x)$, then

$$\int_a^b f(x)\, dx = F(b) - F(a).$$

If $Q'(t)$ is a rate of change, then

$$\int_a^b Q'(t)\, dt = Q(b) - Q(a).$$

If

$$F(x) = \int_a^x f(t)\, dt,$$

then

$$F''(x) = f'(x).$$

Exercises

Part A

Use FTC Part 1 to find the derivative.

1. $$\frac{d}{dx}\left(\int_0^x (t^4 - 3t + 1)\,dt\right)$$

2. $$\frac{d}{dx}\left(\int_2^x \sin t\,dt\right)$$

3. $$\frac{d}{dx}\left(\int_{-1}^x e^t\,dt\right)$$

4. $$\frac{d}{dx}\left(\int_1^x \sqrt{t+4}\,dt\right)$$

Part B

Evaluate the following integrals.

1. $$\int_1^4 x^2\,dx$$

2. $$\int_0^2 (3x+1)\,dx$$

3. $$\int_1^3 \sqrt{x}\,dx$$

4. $$\int_0^\pi \sin x\,dx$$

5. $$\int_{\pi/4}^{\pi/2} \sec^2\theta\,d\theta$$

6. $$\int_1^4 \frac{1}{x^3}\,dx$$

7.
$$\int_0^2 (x+1)(x-2)\,dx$$

Apply the Net Change Theorem.

8. A particle has velocity
$$v(t) = 2t + 1.$$
Find the change in position from $t = 0$ to $t = 4$.

9. A tank is filled at the rate
$$r(t) = 5 - t.$$
Find the total amount added during the first 3 minutes.

Part C

Find the derivative.

1.
$$g(x) = \int_0^x \sqrt{t + t^3}\,dt$$

2.
$$g(x) = \int_1^x \cos(t^2)\,dt$$

3.
$$R(y) = \int_y^2 t^3 \sin t\,dt$$

4.
$$h(x) = \int_2^{1/x} \sin^4 t\,dt$$

5.
$$h(x) = \int_1^{\sqrt{x}} \frac{t^2}{t^4+1}\,dt$$

6.
$$y = \int_1^{3x+2} \frac{t}{1+t^3}\,dt$$

7.
$$y = \int_0^{x^2} \cos^2 \theta \, d\theta$$

8.
$$y = \int_{\sqrt{x}}^{\pi/4} \theta \tan \theta \, d\theta$$

9.
$$y = \int_{\sin x}^{1} \sqrt{1+t^2} \, dt$$

10.
$$g(x) = \int_{2x}^{3x} \frac{u^2 - 1}{u^2 + 1} \, du$$

11.
$$g(x) = \int_{1-2x}^{1+2x} t \sin t \, dt$$

12.
$$h(x) = \int_{\sqrt{x}}^{x^3} \cos(t^2) \, dt$$

Part D

1. A particle has velocity

$$v(t) = 2t + 1.$$

Find the displacement from $t = 0$ to $t = 5$.

2. A particle has velocity

$$v(t) = t^2 - 4.$$

Find the change in position from $t = 0$ to $t = 3$.

3. A population grows according to

$$P'(t) = 200e^{0.05t}.$$

Find the change in population during the first 10 years.

4. Let

$$F(x) = \int_0^x (1 - t^2)\cos^2(t)\,dt.$$

Determine where F is increasing.

5. Let

$$F(x) = \int_1^x f(t)\,dt.$$

Explain how the graph of f can be used to determine where F is concave upward.

6. Evaluate

$$\lim_{n\to\infty} \sum_{i=1}^{n} \left(\frac{i}{n}\right)^3 \frac{1}{n}.$$

7. Evaluate

$$\lim_{n\to\infty} \sum_{i=1}^{n} \sin\left(\frac{i}{n}\right) \frac{1}{n}.$$

8. Evaluate

$$\lim_{n\to\infty} \sum_{i=1}^{n} \left(1 + \frac{i}{n}\right)^2 \frac{1}{n}.$$

9. Find

$$\frac{d}{dx}\left(\int_{x^2}^{x^3} \frac{1}{1+t^2}\,dt\right).$$

10. Suppose

$$F(x) = \int_0^x f(t)\,dt$$

and the graph of f crosses the x-axis at $x = 2$ and $x = 5$.

Explain what can be said about the critical points of F.

Chapter 22

Indefinite Integrals and the Net Change Theorem

22.1. Indefinite Integrals and the Net Change Theorem

Brief Description

In Chapter 19, we used definite integrals to compute areas and accumulated quantities. In this section, we introduce the concept of an **indefinite integral**, which is closely related to derivatives.

The main question is:

Given a function $f(x)$, can we find a function whose derivative is $f(x)$?

Such functions are called **antiderivatives**. Indefinite integrals provide a convenient notation for antiderivatives and are one of the fundamental tools of calculus.

We will also develop a table of basic integration formulas that can be used to evaluate many integrals quickly.

22.1.1 Antiderivatives (Revisited)

Antiderivative

A function F is called an **antiderivative** of a function f on an interval I if

$$F'(x) = f(x)$$

for every x in I.

In other words, an antiderivative reverses the process of differentiation.

Example 22.1.
Find an antiderivative of

$$f(x) = 6x.$$

Solution.
Since

$$\frac{d}{dx}(3x^2) = 6x,$$

an antiderivative is

$$F(x) = 3x^2.$$

$$\boxed{F(x) = 3x^2}$$

The General Antiderivative

Notice that

$$\frac{d}{dx}(3x^2) = 6x,$$

but also

$$\frac{d}{dx}(3x^2 + 5) = 6x,$$

and

$$\frac{d}{dx}(3x^2 - 12) = 6x.$$

Any constant disappears when differentiated. This means antiderivatives are not unique.

Theorem 22.1 (General Antiderivative)

Suppose F is an antiderivative of f on an interval I.
Then every antiderivative of f is of the form

$$F(x) + C,$$

where C is an arbitrary constant.
This collection of all antiderivatives is called the **general antiderivative** of f.

The constant C is called the **constant of integration**.

22.2. Indefinite Integrals

Because antiderivatives are so important, we introduce a special notation.

Indefinite Integral

The notation

$$\int f(x)\,dx$$

represents the family of all antiderivatives of f.
Thus,

$$\int f(x)\,dx = F(x) + C$$

means

$$F'(x) = f(x).$$

The symbol

$$\int$$

is called the **integral sign**, and the expression

$$f(x)$$

is called the **integrand**.

The symbol

$$dx$$

indicates the variable of integration.

Example 22.2.
Evaluate

$$\int x^2\,dx.$$

Solution.
We need a function whose derivative is x^2.
Since

$$\frac{d}{dx}\left(\frac{x^3}{3}\right) = x^2,$$

we obtain

$$\boxed{\int x^2\,dx = \frac{x^3}{3} + C}.$$

22.2.1 Properties of Indefinite Integrals

The properties of differentiation lead directly to the corresponding properties of integration.

Linearity of Integration

For functions f and g and any constant c,

$$\int \big(f(x) \pm g(x)\big)\,dx = \int f(x)\,dx \pm \int g(x)\,dx,$$

and

$$\int c\,f(x)\,dx = c\int f(x)\,dx.$$

22.2.2 Basic Indefinite Integrals

Important

Basic Integration Formulas

$$\int k\,dx = kx + C$$

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C, \qquad n \neq -1$$

$$\int \sin x\,dx = -\cos x + C$$

$$\int \cos x\,dx = \sin x + C$$

$$\int \sec^2 x\,dx = \tan x + C$$

$$\int \csc^2 x\,dx = -\cot x + C$$

$$\int \sec x \tan x\,dx = \sec x + C$$

$$\int \csc x \cot x\,dx = -\csc x + C$$

Warning

When evaluating an indefinite integral, always include the constant of integration:

$$+C.$$

Forgetting the constant is one of the most common mistakes in integration.

22.3. The Net Change Theorem

Recall the Fundamental Theorem of Calculus:

$$\int_a^b f(x)\,dx = F(b) - F(a),$$

where $F'(x) = f(x)$.

This theorem has an important interpretation when f represents a rate of change.

Net Change Theorem

If a quantity Q changes at the rate

$$Q'(t) = r(t),$$

then

$$\int_a^b r(t)\,dt = Q(b) - Q(a).$$

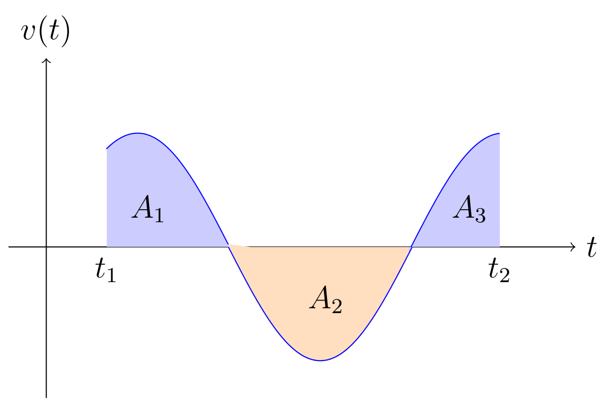
That is:

The integral of a rate of change = The net change in the quantity.

22.3.1 Visual Interpretation

The Net Change Theorem says that integrating a rate accumulates all of the changes that occur over an interval.

The figure below illustrates positive and negative contributions to the integral.



The definite integral equals $\int_{t_1}^{t_2} v(t)\,dt = A_1 - A_2 + A_3$.

Areas above the t-axis contribute positively, while areas below the axis contribute negatively.

22.3.2 Position and Velocity

Suppose $s(t)$ represents the position of an object.

Then

$$s'(t) = v(t),$$

where $v(t)$ is velocity.

Applying the Net Change Theorem,

$$\int_a^b v(t)\,dt = s(b) - s(a).$$

Thus the integral of velocity gives the change in position.

Important

$$\boxed{\int_a^b v(t)\,dt = \text{change in position}}$$

Example 22.3.
A particle moves with velocity

$$v(t) = 3t^2 + 2$$

meters per second.
Find the change in position from $t = 0$ to $t = 2$.
Solution.
By the Net Change Theorem,

$$\Delta s = \int_0^2 (3t^2 + 2)\,dt.$$

First find an antiderivative:

$$\int (3t^2 + 2)\,dt = t^3 + 2t.$$

Now evaluate:

$$\Delta s = \Big[t^3 + 2t\Big]_0^2 = (8 + 4) - 0.$$

$$\boxed{\Delta s = 12 \text{ meters}}$$

22.3.3 Velocity and Acceleration

Recall that

$$a(t) = v'(t).$$

Therefore,

$$\int_a^b a(t)\,dt = v(b) - v(a).$$

The integral of acceleration gives the change in velocity.

Important

$$\boxed{\int_a^b a(t)\,dt = \text{change in velocity}}$$

Example 22.4.
The acceleration of a car is

$$a(t) = 4t$$

meters per second squared.
Find the change in velocity from $t = 1$ to $t = 3$.
Solution.
Using the Net Change Theorem,

$$\Delta v = \int_1^3 4t\,dt.$$

An antiderivative is

$$2t^2.$$

Therefore,

$$\Delta v = \left[2t^2\right]_1^3 = 2(9) - 2(1) = 18 - 2.$$

$$\boxed{\Delta v = 16 \text{ m/s}}$$

22.3.4 A Note on Units

A definite integral inherits units from both the function and the variable.

Units of Integration

If
$$y = f(x),$$
then
$$\int_a^b f(x)\,dx$$
has units
$$(\text{units of } f) \times (\text{units of } x).$$

For example:

- Velocity (meters/second) $\times$ time (seconds) $=$ meters.
- Force (newtons) $\times$ distance (meters) $=$ joules.
- Power (megawatts) $\times$ time (hours) $=$ megawatt-hours.

Example 22.5.
A power plant generates electricity at the rate

$$P(t) = 50$$

megawatts for 4 hours.
Interpret
$$\int_0^4 P(t)\,dt.$$

Solution.
Since power is measured in megawatts and time is measured in hours,

$$\int_0^4 P(t)\,dt$$

measures
$$(\text{megawatts})(\text{hours}) = \text{megawatt-hours}.$$

Evaluating,
$$\int_0^4 50\,dt = 50(4) = 200.$$
$$\boxed{200 \text{ megawatt-hours}}$$

This represents the total energy produced during the four-hour period.

Additional Examples

Example 22.6.
Evaluate

$$\int \left(5 + \frac{2}{3}x^2 + \frac{3}{4}x^3\right)\, dx.$$

Solution.
Integrate each term separately:

$$\int 5\, dx = 5x,$$

$$\int \frac{2}{3}x^2\, dx = \frac{2}{3} \cdot \frac{x^3}{3} = \frac{2x^3}{9},$$

$$\int \frac{3}{4}x^3\, dx = \frac{3}{4} \cdot \frac{x^4}{4} = \frac{3x^4}{16}.$$

Therefore,

$$\boxed{\int \left(5 + \frac{2}{3}x^2 + \frac{3}{4}x^3\right)\, dx = 5x + \frac{2x^3}{9} + \frac{3x^4}{16} + C}$$

Example 22.7.
Evaluate

$$\int (u+4)\,(2u+1)\, du.$$

Solution.
First expand:

$$(u+4)(2u+1) = 2u^2 + 9u + 4.$$

Therefore,

$$\int (u+4)(2u+1)\, du = \int (2u^2 + 9u + 4)\, du.$$

Integrating term-by-term,

$$= \frac{2u^3}{3} + \frac{9u^2}{2} + 4u + C.$$

$$\boxed{\int (u+4)(2u+1)\,du = \frac{2u^3}{3} + \frac{9u^2}{2} + 4u + C}$$

Example 22.8.
Evaluate

$$\int \left(2 + \tan^2\theta\right)\,d\theta.$$

Solution.
Recall the identity

$$\tan^2\theta = \sec^2\theta - 1.$$

Thus,

$$2 + \tan^2\theta = 1 + \sec^2\theta.$$

Therefore,

$$\int (2 + \tan^2\theta)\,d\theta = \int (1 + \sec^2\theta)\,d\theta.$$

By integrating we get,

$$= \theta + \tan\theta + C.$$

$$\boxed{\int (2 + \tan^2\theta)\,d\theta = \theta + \tan\theta + C}$$

Example 22.9.
Evaluate

$$\int_1^2 \left(4x^3 - 3x^2 + 2x\right)\,dx.$$

Solution.
An antiderivative is

$$F(x) = x^4 - x^3 + x^2.$$

Using the Fundamental Theorem of Calculus,

$$\int_1^2 \left(4x^3 - 3x^2 + 2x\right)\,dx = F(2) - F(1).$$

Thus,

$$F(2) = 16 - 8 + 4 = 12,$$

$$F(1) = 1 - 1 + 1 = 1.$$

Hence,

$$\boxed{\int_1^2 \left(4x^3 - 3x^2 + 2x\right)\, dx = 11}$$

22.4. Summary

Important

Key Ideas

- A function F is an antiderivative of f if

$$F'(x) = f(x).$$

- The indefinite integral represents the family of all antiderivatives:

$$\int f(x)\, dx = F(x) + C.$$

- Always include the constant of integration C when evaluating an indefinite integral.
- Power Rule for Integration:

$$\int x^n\, dx = \frac{x^{n+1}}{n+1} + C, \qquad n \neq -1.$$

- Net Change Theorem:

$$\int_a^b r(t)\, dt = Q(b) - Q(a).$$

- The integral of velocity gives change in position.
- The integral of acceleration gives change in velocity.
- Units of a definite integral are

$$(\text{units of function}) \times (\text{units of variable}).$$

Exercises

Evaluate the indefinite integral.

1.
$$\int (x^{1.3} + 7x^{2.5})\,dx$$

2.
$$\int \sqrt[4]{x^5}\,dx$$

3.
$$\int \left(5 + \frac{2}{3}x^2 + \frac{3}{4}x^3\right)\,dx$$

4.
$$\int \left(u^6 - 2u^5 - u^3 + \frac{2}{7}\right)\,du$$

5.
$$\int (u+4)(2u+1)\,du$$

6.
$$\int (2 + \tan^2\theta)\,d\theta$$

7.
$$\int \sec t(\sec t + \tan t)\,dt$$

8.
$$\int \frac{1 - \sin^3 t}{\sin^2 t}\,dt$$

9.
$$\int \frac{\sin(2x)}{\sin x}\,dx$$

Evaluate the definite integral.

10.
$$\int_{-2}^{3} (x^2 - 3)\,dx$$

11.
$$\int_{1}^{2} (4x^3 - 3x^2 + 2x)\,dx$$

12.
$$\int_{-2}^{0} \left(\frac{1}{2}t^4 + \frac{1}{4}t^3 - t\right)\,dt$$

13.
$$\int_0^3 (1 + 6w^2 - 10w^4)\, dw$$

14.
$$\int_0^2 (2x - 3)(4x^2 + 1)\, dx$$

15.
$$\int_1^4 \sqrt{\frac{5}{x}}\, dx$$

16.
$$\int_1^8 \left(\frac{2}{\sqrt[3]{w}} - \sqrt[3]{w} \right)\, dw$$

17.
$$\int_1^4 \sqrt{t}(1 + t)\, dt$$

18.
$$\int_0^{\pi/4} \sec\theta \tan\theta\, d\theta$$

19.
$$\int_0^{\pi/4} \frac{1 + \cos^2\theta}{\cos^2\theta}\, d\theta$$

Answer the following conceptual questions.

20. If $w'(t)$ is the rate of growth of a child (in pounds per year), what does
$$\int_5^{10} w'(t)\, dt$$
represent?

21. If $f(x)$ is the slope of a trail at a distance of x miles from the start, what does
$$\int_3^5 f(x)\, dx$$
represent?

22. If x is measured in meters and $f(x)$ is measured in newtons, what are the units of
$$\int_0^{100} f(x)\, dx?$$

23. If x is measured in feet and $a(x)$ is measured in pounds per foot, determine the units of
$$\frac{da}{dx} \qquad \text{and} \qquad \int_2^8 a(x)\, dx.$$

Chapter 23

The Substitution Rule

23.1. The Substitution Rule

Brief Description

In Chapter 22, we learned how to evaluate many integrals using the Fundamental Theorem of Calculus. However, many integrands are not written in a form whose antiderivative is immediately recognizable. For example,

$$\int 2x(x^2+1)^5\,dx$$

is more complicated than

$$\int u^5\,du.$$

The **Substitution Rule** allows us to simplify such integrals by introducing a new variable. This rule is the reverse process of the **Chain Rule** for differentiation.

Recall that if

$$F'(u) = f(u)$$

and

$$u = g(x),$$

Then, by the Chain Rule, we have:

$$\frac{d}{dx}F(g(x)) = F'(g(x))g'(x) = f(g(x))g'(x).$$

Reversing this process leads to the Substitution Rule.

Important

The Substitution Rule is often called **u-substitution**. Its goal is to transform a complicated integral into a simpler one by replacing part of the integrand with a new variable u.

A Tip

One of the most common questions students ask is:

"How do I know what to choose for u?"

With practice, the answer becomes more intuitive.
A good first strategy is to look for an "inside" function whose derivative also appears elsewhere in the integrand.
Although this rule is not always sufficient, it works surprisingly often.

23.1.1 The Substitution Rule

Theorem 23.1 (Substitution Rule)

Suppose $u = g(x)$ is differentiable and f is continuous. Then

$$\int f(g(x))g'(x)\,dx = \int f(u)\,du,$$

where

$$u = g(x) \qquad \text{and} \qquad du = g'(x)\,dx.$$

Important

A useful way to remember the rule is:

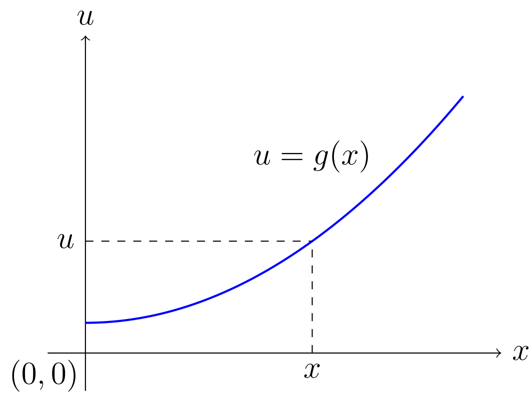
$$u = g(x) \quad \Longrightarrow \quad du = g'(x)\,dx.$$

Then replace every occurrence of x and dx with expressions involving u.

23.1.2 Geometric Interpretation

The substitution rule changes variables without changing the accumulated area represented by the integral.

The variable u is simply a new coordinate system that may make the integral easier to evaluate.



Example 23.1.

Evaluate

$$\int 2x(x^2+1)^5\,dx.$$

Solution.

Let

$$u = x^2 + 1.$$

Then

$$du = 2x\,dx.$$

Substituting,

$$\int 2x(x^2+1)^5\,dx = \int u^5\,du.$$

Now integrate:

$$\int u^5\,du = \frac{u^6}{6} + C.$$

Substituting back,

$$\boxed{\int 2x(x^2+1)^5\,dx = \frac{(x^2+1)^6}{6} + C}.$$

Example 23.2.
Evaluate

$$\int \cos(3x)\, dx.$$

Solution.
Let

$$u = 3x.$$

Then

$$du = 3\, dx, \qquad \text{and therefore} \qquad dx = \frac{1}{3} du.$$

Thus,

$$\int \cos(3x)\, dx = \frac{1}{3} \int \cos u\, du.$$

By integrating we then get,

$$= \frac{1}{3} \sin u + C.$$

Substituting back,

$$\boxed{\int \cos(3x)\, dx = \frac{1}{3} \sin(3x) + C}.$$

Example 23.3.
Evaluate

$$\int \frac{x}{x^2 + 4}\, dx.$$

Solution.
Let

$$u = x^2 + 4.$$

Then

$$du = 2x\, dx, \qquad x\, dx = \frac{1}{2}\, du.$$

Hence

$$\int \frac{x}{x^2+4}\,dx = \frac{1}{2}\int \frac{1}{u}\,du.$$

Using

$$\int \frac{1}{u}\,du = \ln|u| + C,$$

we obtain

$$= \frac{1}{2}\ln|u| + C.$$

Substituting back,

$$\boxed{\int \frac{x}{x^2+4}\,dx = \frac{1}{2}\ln(x^2+4) + C}.$$

23.1.3 Strategy for u-Substitution

When using substitution:

1. Look for a function inside another function.
2. Choose
$$u = \text{(inside function)}.$$
3. Compute du.
4. Rewrite the entire integral in terms of u.
5. Integrate with respect to u.
6. Substitute back to the original variable.

Warning

Always substitute back to the original variable unless the problem specifically asks for the answer in terms of u.

23.1.4 Recognizing When to Use Substitution

One of the most important skills in integration is recognizing when a substitution is appropriate.

A substitution is often useful when the integrand contains:

- A composite function, such as

$$(3x+1)^5, \qquad \sin(x^2), \qquad \sqrt{1+x^3},$$

- A function together with its derivative,

$$(2x)(x^2+1)^7,$$

$$\cos(5x),$$

$$\frac{x}{x^2+4},$$

- A radical expression whose inside function has a derivative elsewhere in the integrand.

Important

A good first guess is:

$$u = \text{the inside function.}$$

This often transforms a difficult integral into a familiar one.

23.1.5 Polynomial Substitutions

Example 23.4.
Evaluate

$$\int x(x^2+3)^4\,dx.$$

Solution.
Let

$$u = x^2+3.$$

Then

$$du = 2x\,dx, \qquad x\,dx = \frac{1}{2}\,du.$$

Substituting,

$$\int x(x^2+3)^4\,dx = \frac{1}{2}\int u^4\,du.$$

Integrating,

$$= \frac{1}{2}\cdot\frac{u^5}{5} + C = \frac{u^5}{10} + C.$$

Substituting back,

$$\boxed{\int x(x^2+3)^4\,dx = \frac{(x^2+3)^5}{10} + C}.$$

23.1.6 Radical Substitutions

Example 23.5.
Evaluate

$$\int x^2\sqrt{x^3+1}\,dx.$$

Solution.
Let

$$u = x^3 + 1.$$

Then

$$du = 3x^2\,dx, \qquad x^2\,dx = \frac{1}{3}\,du.$$

Therefore,

$$\int x^2\sqrt{x^3+1}\,dx = \frac{1}{3}\int u^{1/2}\,du.$$

Using the Power Rule,

$$= \frac{1}{3}\left(\frac{u^{3/2}}{3/2}\right) + C = \frac{2}{9}u^{3/2} + C.$$

Substituting back,

$$\boxed{\int x^2\sqrt{x^3+1}\,dx = \frac{2}{9}(x^3+1)^{3/2} + C}.$$

23.1.7 Trigonometric Substitutions

Many trigonometric integrals can be evaluated using a simple substitution.

Example 23.6.
Evaluate
$$\int \sin^2\theta\,\cos\theta\,d\theta.$$

Solution.
Let
$$u = \sin\theta.$$
Then
$$du = \cos\theta\,d\theta.$$
Substituting,
$$\int \sin^2\theta\,\cos\theta\,d\theta = \int u^2\,du.$$
Integrating,
$$= \frac{u^3}{3} + C.$$
Replacing u,
$$\boxed{\int \sin^2\theta\,\cos\theta\,d\theta = \frac{\sin^3\theta}{3} + C}.$$

Example 23.7.
Evaluate
$$\int \sec(3x)\tan(3x)\,dx.$$

Solution.
Let
$$u = 3x.$$
Then
$$du = 3\,dx, \qquad dx = \frac{1}{3}\,du.$$
Hence,
$$\int \sec(3x)\tan(3x)\,dx = \frac{1}{3}\int \sec u\tan u\,du.$$

Since

$$\int \sec u \tan u \, du = \sec u + C,$$

we obtain

$$= \frac{1}{3} \sec u + C.$$

Substituting back,

$$\boxed{\int \sec(3x) \tan(3x) \, dx = \frac{1}{3} \sec(3x) + C}.$$

23.1.8 Exponential Substitutions

Example 23.8.
Evaluate

$$\int e^{5x} \, dx.$$

Solution.
Let

$$u = 5x.$$

Then

$$du = 5 \, dx, \qquad dx = \frac{1}{5} \, du.$$

Therefore,

$$\int e^{5x} \, dx = \frac{1}{5} \int e^{u} \, du.$$

Integrating,

$$= \frac{1}{5} e^{u} + C.$$

Replacing u,

$$\boxed{\int e^{5x} \, dx = \frac{1}{5} e^{5x} + C}.$$

23.1.9 Common Mistakes

Warning

Do not substitute only part of the integral.
For example,

$$\int x(x^2+1)^3\,dx$$

with

$$u = x^2 + 1$$

requires that $x\,dx$ also be rewritten in terms of u.

Warning

Do not leave both u and x in the same integral.
For example,

$$\int (x^2+1)^3\,du$$

is incorrect because the integral must be entirely in terms of u.

Warning

After integrating with respect to u, substitute back to the original variable unless instructed otherwise.

23.2. The Substitution Rule for Definite Integrals

In Chapter 22, we learned that substitution can simplify indefinite integrals. The same idea works for definite integrals, but there is one important difference: the limits of integration must also be changed.

Changing the limits allows us to complete the entire calculation in terms of the new variable without converting back to the original variable.

Theorem 23.2 (Substitution Rule for Definite Integrals)

Suppose g is differentiable on $[a, b]$, g' is continuous, and f is continuous on the range of g. Then

$$\int_a^b f(g(x))\,g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du.$$

Important

When using substitution in a definite integral:

1. Choose a substitution $u = g(x)$.
2. Compute $du = g'(x)\,dx$.
3. Change the limits from x-values to u-values.
4. Rewrite the entire integral in terms of u.
5. Evaluate the resulting definite integral.

A Step-by-Step Example

Example 23.9.
Evaluate

$$\int_0^1 (2x+1)^5\,dx.$$

Solution.
Let

$$u = 2x + 1.$$

Then

$$du = 2\,dx, \qquad dx = \frac{1}{2}\,du.$$

Next, change the limits:

$$x = 0 \implies u = 1,$$

$$x = 1 \implies u = 3.$$

Therefore,

$$\int_0^1 (2x+1)^5\,dx = \frac{1}{2}\int_1^3 u^5\,du.$$

Now evaluate:

$$= \frac{1}{2}\left[\frac{u^6}{6}\right]_1^3 = \frac{1}{12}(3^6 - 1^6).$$

Since

$$3^6 = 729,$$

we obtain

$$= \frac{1}{12}(729 - 1) = \frac{728}{12} = \frac{182}{3}.$$

$$\boxed{\int_0^1 (2x+1)^5\,dx = \frac{182}{3}}$$

Example with a Radical

Example 23.10.
Evaluate

$$\int_0^3 x\sqrt{x^2+1}\,dx.$$

Solution.
Let

$$u = x^2 + 1.$$

Then

$$du = 2x\,dx, \qquad x\,dx = \frac{1}{2}\,du.$$

Change the limits:

$$x = 0 \Rightarrow u = 1,$$

$$x = 3 \Rightarrow u = 10.$$

Therefore,

$$\int_0^3 x\sqrt{x^2+1}\,dx = \frac{1}{2}\int_1^{10} u^{1/2}\,du.$$

Integrating,

$$= \frac{1}{2}\left[\frac{2}{3}u^{3/2}\right]_1^{10} = \frac{1}{3}\left[u^{3/2}\right]_1^{10}.$$

Thus,

$$= \frac{1}{3}\left(10^{3/2} - 1\right).$$

Since

$$10^{3/2} = 10\sqrt{10},$$

we obtain

$$\boxed{\int_0^3 x\sqrt{x^2+1}\,dx = \frac{10\sqrt{10}-1}{3}}.$$

Example with Trigonometric Functions

Example 23.11.
Evaluate

$$\int_0^{\pi/2} \sin x\,\sqrt{1+\cos x}\,dx.$$

Solution.
Let

$$u = 1 + \cos x.$$

Then

$$du = -\sin x\,dx.$$

Hence

$$-du = \sin x\,dx.$$

Change the limits:

$$x = 0 \Rightarrow u = 2,$$

$$x = \frac{\pi}{2} \Rightarrow u = 1.$$

Thus,

$$\int_0^{\pi/2} \sin x \sqrt{1 + \cos x}\, dx = -\int_2^1 u^{1/2}\, du.$$

Reversing limits,

$$= \int_1^2 u^{1/2}\, du.$$

Now integrate:

$$= \left[\frac{2}{3} u^{3/2}\right]_1^2.$$

Therefore,

$$= \frac{2}{3}\left(2^{3/2} - 1\right) = \frac{2}{3}\left(2\sqrt{2} - 1\right).$$

$$\boxed{\int_0^{\pi/2} \sin x \sqrt{1 + \cos x}\, dx = \frac{2}{3}(2\sqrt{2} - 1)}$$

23.3. A Geometric Interpretation

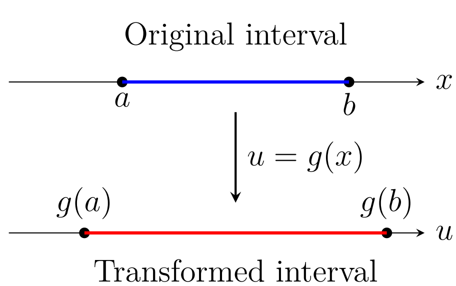
The substitution rule can be viewed as a change of coordinates.

If

$$u = g(x),$$

then the interval in the x-variable is transformed into an interval in the u-variable.

The substitution rule says that the area computed using x-coordinates is exactly the same as the area computed using the new u-coordinates.



23.4. Common Errors

Warning

When changing variables in a definite integral, do not keep the original limits.
Once the integral has been rewritten completely in terms of u, the limits must also be written in terms of u.

Warning

Do not mix variables:

$$\int_1^3 u^2 \, dx$$

is not valid because both u and x appear simultaneously.

Warning

A negative sign often appears when $du = -g'(x)\, dx$.
Be careful when reversing limits.

23.5. Symmetry and Integration

Sometimes a definite integral can be evaluated without finding an antiderivative. This is possible when the integrand possesses symmetry.

Even Functions

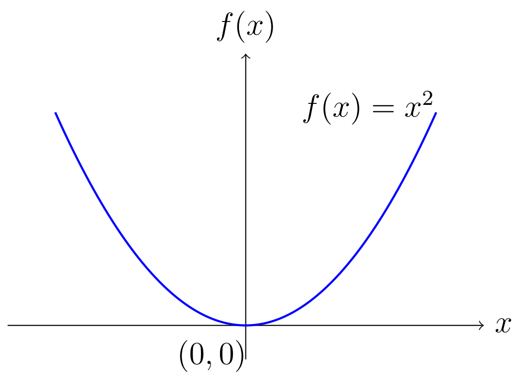
Even Function

A function f is called **even** if

$$f(-x) = f(x)$$

for all x in its domain.

Graphically, an even function is symmetric about the y-axis. An example is $f(x) = x^2$.



Theorem 23.3 (Integral of an Even Function)

If f is even and integrable on $[-a, a]$, then

$$\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx.$$

Example 23.12.

Evaluate

$$\int_{-2}^{2} (x^4 + 1)\,dx.$$

Solution.

Since

$$f(x) = x^4 + 1$$

is even,

$$\int_{-2}^{2} (x^4 + 1)\,dx = 2\int_{0}^{2} (x^4 + 1)\,dx.$$

Therefore,

$$= 2\left[\frac{x^5}{5} + x\right]_0^2.$$

$$= 2\left(\frac{32}{5} + 2\right) = 2\left(\frac{42}{5}\right).$$

$$= \boxed{\frac{84}{5}}.$$

Odd Functions

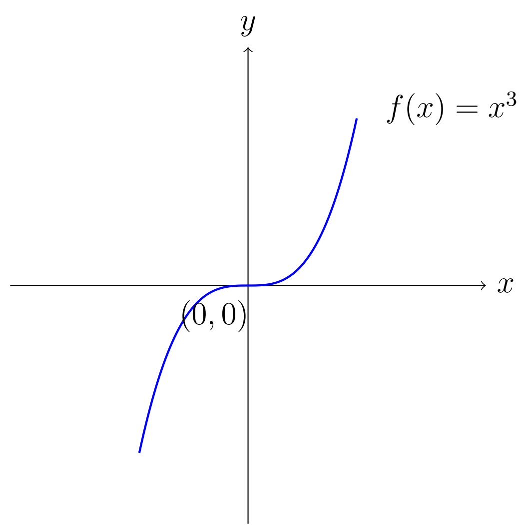
Odd Function

A function f is called **odd** if

$$f(-x) = -f(x)$$

for all x in its domain.

Graphically, an odd function is symmetric about the origin. An example is $f(x) = x^3$.



Theorem 23.4 (Integral of an Odd Function)

If f is odd and integrable on $[-a, a]$, then

$$\int_{-a}^{a} f(x)\,dx = 0.$$

Example 23.13.

Evaluate

$$\int_{-3}^{3} (x^5 + x)\,dx.$$

Solution.

Since

$$f(x) = x^5 + x$$

is odd, we have

$$\boxed{\int_{-3}^{3} (x^5 + x)\,dx = 0}.$$

23.5.1 Combining Symmetry with Substitution

Many definite integrals become easier when symmetry and substitution are used together. Always check:

1. Is the function even?
2. Is the function odd?
3. Does a substitution simplify the integrand?

Example 23.14.

Evaluate

$$\int_0^1 x\sqrt{1+x^2}\,dx.$$

Solution.

Let

$$u = 1 + x^2.$$

Then

$$du = 2x\,dx, \qquad x\,dx = \frac{1}{2}\,du.$$

When $x = 0$, we have

$$u = 1.$$

When $x = 1$, we have

$$u = 2.$$

Thus,

$$\int_0^1 x\sqrt{1+x^2}\,dx = \frac{1}{2}\int_1^2 u^{1/2}\,du.$$

$$= \frac{1}{2}\left[\frac{2}{3}u^{3/2}\right]_1^2.$$

$$= \frac{1}{3}\left(2^{3/2} - 1\right).$$

$$= \boxed{\frac{1}{3}(2\sqrt{2} - 1)}.$$

23.6. Summary

Important

For indefinite integrals:

$$u = g(x), \qquad du = g'(x)\,dx.$$

Then

$$\int f(g(x))g'(x)\,dx = \int f(u)\,du.$$

For definite integrals:

$$\int_a^b f(g(x))g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du.$$

When using substitution:

1. Choose a suitable u.
2. Compute du.
3. Rewrite the entire integral.
4. Change limits if the integral is definite.
5. Integrate.

Exercises

Part A

Evaluate each integral using substitution.

1. $$\int \cos(2x)\,dx$$

2. $$\int x(2x^2+3)^4\,dx$$

3. $$\int x^2\sqrt{x^3+1}\,dx$$

4. $$\int \sin^4\theta\cos\theta\,d\theta$$

5. $$\int \frac{x^3}{(x^4-5)^2}\,dx$$

6. $$\int \sqrt{2t+1}\,dt$$

7. $$\int x\sqrt{1-x^2}\,dx$$

8. $$\int x^2\sin(x^3)\,dx$$

9. $$\int (1-2x)^9\,dx$$

10. $$\int \sin t\,\sqrt{1+\cos t}\,dt$$

Part B

Evaluate each integral using the Substitution Rule for Definite Integrals.

1. $$\int_0^1 \cos(2x)\,dx$$

2.
$$\int_0^2 x(2x^2+3)^4\,dx$$

3.
$$\int_0^1 x^2\sqrt{x^3+1}\,dx$$

4.
$$\int_0^{\pi/2} \sin^3\theta\cos\theta\,d\theta$$

5.
$$\int_1^2 \frac{x^3}{(x^4-5)^2}\,dx$$

6.
$$\int_0^3 \sqrt{2t+1}\,dt$$

7.
$$\int_0^1 x^2\sin(x^3)\,dx$$

8.
$$\int_0^{\sqrt{\pi}} x\cos(x^2)\,dx$$

9.
$$\int_0^{\pi/6} \frac{\sin t}{\cos^2 t}\,dt$$

10.
$$\int_0^1 (3x-1)^{50}\,dx$$

Part C

Evaluate each integral using substitution.

1.
$$\int (4x+1)^8\,dx$$

2.
$$\int x(x^2+9)^6\,dx$$

3.
$$\int \frac{x}{x^2+16}\,dx$$

4.
$$\int x^2 \cos(x^3)\, dx$$

5.
$$\int \sin(5x)\, dx$$

6.
$$\int \sec^2(4x)\, dx$$

7.
$$\int e^{7x}\, dx$$

8.
$$\int \frac{1}{\sqrt{3x+2}}\, dx$$

9.
$$\int_0^2 x(x^2+1)^3\, dx$$

10.
$$\int_0^{\pi/2} \sin x \sqrt{1+\cos x}\, dx$$

11.
$$\int_1^2 \frac{x^3}{(x^4+1)^2}\, dx$$

12.
$$\int_0^1 x^2 e^{x^3}\, dx$$

Use symmetry whenever possible.

13.
$$\int_{-2}^2 (x^6+1)\, dx$$

14.
$$\int_{-3}^3 (x^7+x)\, dx$$

15.
$$\int_{-\pi}^{\pi} \sin x\, dx$$

16.
$$\int_{-1}^1 \cos x\, dx$$

17. Determine whether

$$f(x) = x^8 - 3x^4 + 2$$

is even, odd, or neither.

18. Determine whether

$$f(x) = x^5 - 2x$$

is even, odd, or neither.

Chapter 24

Areas Between Curves

24.1. Areas Between Curves

Brief Description

In Section 4.1 we learned that a definite integral can be used to find the area under a curve.

In many applications, however, we are interested in finding the area enclosed by two curves rather than the area between a curve and the x-axis.

The basic idea is simple. If $f(x)$ lies above $g(x)$ on an interval $[a, b]$, then the height of a typical approximating rectangle is

$$f(x) - g(x),$$

and its width is

$$\Delta x.$$

Adding the areas of these rectangles and taking a limit leads to a formula for the area between two curves.

Throughout this section we will learn how to:

- Find areas between curves using vertical rectangles.
- Handle curves that intersect.
- Find areas using horizontal rectangles.
- Decide whether to integrate with respect to x or y.

Area Between Two Curves

Suppose f and g are continuous functions on $[a, b]$ and

$$f(x) \geq g(x)$$

for all $x \in [a, b]$.

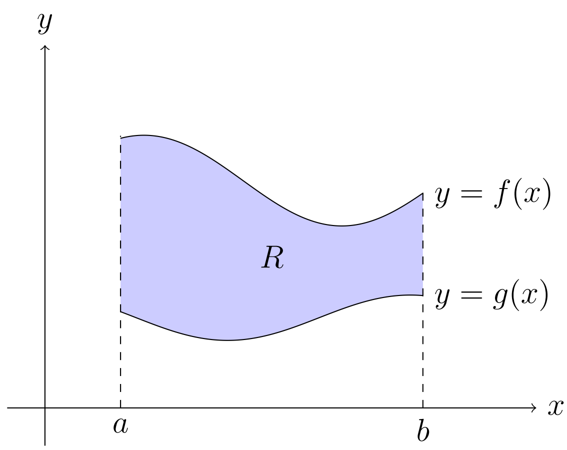
The region enclosed by the curves

$$y = f(x), \qquad y = g(x),$$

and the vertical lines

$$x = a, \qquad x = b$$

is shown below.



A typical approximating rectangle has

$$\text{height} = f(x) - g(x)$$

and

$$\text{width} = \Delta x.$$

Therefore,

$$\text{Area of rectangle} \approx [f(x) - g(x)]\, \Delta x.$$

Adding all such rectangles and taking a limit gives the following theorem.

Theorem 24.1 (Area Between Two Curves)

Suppose f and g are continuous on $[a, b]$ and

$$f(x) \geq g(x)$$

for every $x \in [a, b]$.
Then the area of the region enclosed by the curves is

$$A = \int_a^b [f(x) - g(x)]\, dx.$$

Equivalently,

$$\boxed{A = \int_a^b (\text{top} - \text{bottom})\, dx}$$

24.2. Strategy for Finding Areas

When finding the area between two curves:

1. Sketch the region.
2. Determine which function is on top.
3. Find the points of intersection if needed.
4. Compute

$$A = \int_a^b (\text{top} - \text{bottom})\, dx.$$

Example 24.1.
Find the area of the region bounded by

$$y = x + 2 \quad \text{and} \quad y = x^2$$

between their points of intersection.
Solution.
First, find the points of intersection:

$$x + 2 = x^2.$$

Thus,

$$x^2 - x - 2 = 0,$$

so

$$(x-2)(x+1) = 0.$$

Therefore, the curves intersect at

$$x = -1 \qquad \text{and} \qquad x = 2.$$

On the interval $[-1, 2]$, the line $y = x + 2$ lies above the parabola $y = x^2$. Hence, the area is

$$A = \int_{-1}^{2} \left[(x+2) - x^2\right] \, dx.$$

Evaluate:

$$A = \left[\frac{x^2}{2} + 2x - \frac{x^3}{3}\right]_{-1}^{2}.$$

Therefore,

$$A = \left(2 + 4 - \frac{8}{3}\right) - \left(\frac{1}{2} - 2 + \frac{1}{3}\right).$$

$$A = \frac{10}{3} + \frac{7}{6} = \frac{27}{6} = \frac{9}{2}.$$

Thus, the area of the region is

$$\boxed{\frac{9}{2}}.$$

Example 24.2.

Find the area of the region bounded by

$$y = 4 - x \quad \text{and} \quad y = 1 \quad \text{for} \quad 0 \le x \le 3.$$

Solution.

On the interval $0 \le x \le 3$, the line

$$y = 4 - x$$

lies above the line

$$y = 1.$$

Therefore, the area is

$$A = \int_0^3 [(4 - x) - 1]\, dx = \int_0^3 (3 - x)\, dx.$$

By evaluating we get,

$$A = \left[3x - \frac{x^2}{2}\right]_0^3 = 9 - \frac{9}{2} = \frac{9}{2}.$$

Thus, the area of the region is

$$\boxed{\frac{9}{2}}.$$

Warning

Area is always positive.
When using

$$A = \int_a^b [f(x) - g(x)]\, dx,$$

be sure that $f(x)$ is the upper function and $g(x)$ is the lower function throughout the interval.
If the curves intersect, the interval may need to be split into separate integrals.

24.3. When Curves Intersect

The formula

$$A = \int_a^b [f(x) - g(x)]\, dx$$

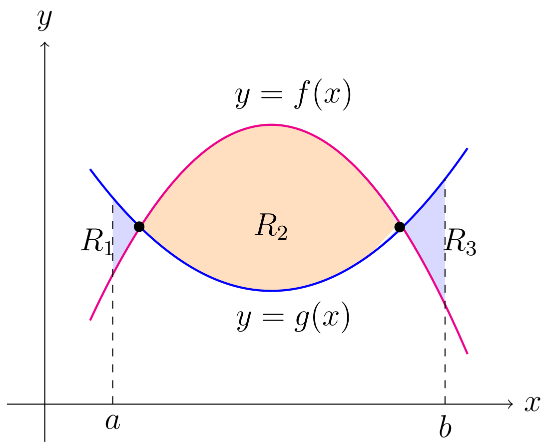
works only when the same function remains above the other throughout the interval.

However, many regions are bounded by curves that intersect. In such cases, the upper and lower functions may change.

Consider the region enclosed by the curves

$$y = f(x) \qquad \text{and} \qquad y = g(x)$$

shown below.



Notice that the upper function changes as we move from left to right. Therefore, we must split the region into pieces and compute each area separately.

A convenient formula that always works is:

Important

Area Between Intersecting Curves

The area between the curves

$$y = f(x) \qquad \text{and} \qquad y = g(x)$$

from $x = a$ to $x = b$ is

$$\boxed{A = \int_a^b |f(x) - g(x)|\, dx}$$

In practice, we usually:

1. Find the intersection points.
2. Determine which function is on top on each interval.
3. Split the integral accordingly.
4. Add the resulting areas.

Example 24.3.

Find the area enclosed by

$$y = 2x \qquad \text{and} \qquad y = x^2 - 4x.$$

Solution.
First, find the points of intersection:

$$2x = x^2 - 4x.$$

By simplifying we get

$$x(x-6) = 0.$$

Therefore, the curves intersect at

$$x = 0 \qquad \text{and} \qquad x = 6.$$

On the interval $[0, 6]$, the line $y = 2x$ lies above the parabola $y = x^2 - 4x$. Therefore, the enclosed area is

$$A = \int_0^6 \left[2x - (x^2 - 4x)\right] \, dx.$$

Simplifying,

$$A = \int_0^6 (6x - x^2) \, dx.$$

Hence,

$$A = \left[3x^2 - \frac{x^3}{3}\right]_0^6.$$

$$A = \left(3(6)^2 - \frac{6^3}{3}\right) - 0 = 108 - 72 = 36.$$

Thus, the area enclosed by the curves is

$$\boxed{36}.$$

Example 24.4.
Find the area enclosed by

$$y = x^2$$

and

$$y = 2 - x.$$

Solution.
First, find the points of intersection:

$$x^2 = 2 - x.$$

Thus,

$$x^2 + x - 2 = 0,$$

so

$$(x+2)(x-1) = 0.$$

Therefore, the curves intersect at

$$x = -2 \qquad \text{and} \qquad x = 1.$$

On the interval $[-2, 1]$, the line $y = 2 - x$ lies above the parabola $y = x^2$. Therefore, the enclosed area is

$$A = \int_{-2}^{1} \left[(2-x) - x^2\right] \, dx.$$

Hence,

$$A = \left[2x - \frac{x^2}{2} - \frac{x^3}{3}\right]_{-2}^{1}.$$

$$A = \left(2 - \frac{1}{2} - \frac{1}{3}\right) - \left(-4 - 2 + \frac{8}{3}\right).$$

$$A = \frac{7}{6} + \frac{10}{3} = \frac{27}{6} = \frac{9}{2}.$$

Thus, the area enclosed by the curves is

$$\boxed{\frac{9}{2}}.$$

Important

When curves intersect, the safest approach is:

$$\boxed{A = \int (\text{top} - \text{bottom}) \, dx}$$

on each subinterval determined by the intersection points.

24.4. A Geometric Interpretation of Absolute Value

Recall that a definite integral measures signed area. Regions above the x-axis contribute positively, while regions below the x-axis contribute negatively.

Similarly,

$$\int_a^b [f(x) - g(x)]\, dx$$

computes a signed area between the curves.

Using absolute value ensures that every piece contributes positively:

$$A = \int_a^b |f(x) - g(x)|\, dx.$$

Thus, the area is always nonnegative.

24.4.1 Integrating with Respect to y

Sometimes, vertical rectangles are not the easiest choice.

For some regions, it is better to use **horizontal rectangles**. In that case, we write the curves as functions of y, such as

$$x = f(y) \qquad \text{and} \qquad x = g(y).$$

If the region lies between

$$y = c \qquad \text{and} \qquad y = d,$$

then a typical horizontal rectangle has

$$\text{width} = x_R - x_L,$$

where x_R is the right boundary and x_L is the left boundary.

Its height is

$$\Delta y.$$

Therefore,

$$\text{Area of rectangle} \approx (x_R - x_L)\Delta y.$$

Taking the limit gives the area formula with respect to y.

Important

Area Using Horizontal Rectangles
If a region is bounded on the right by

$$x = x_R(y)$$

and on the left by

$$x = x_L(y),$$

for

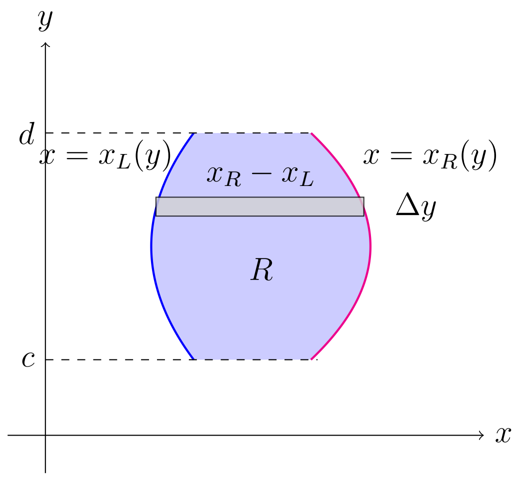
$$c \le y \le d,$$

then the area is

$$A = \int_c^d [x_R(y) - x_L(y)]\,dy.$$

Equivalently,

$$\boxed{A = \int_c^d (\text{right} - \text{left})\,dy}$$



24.4.2 Choosing dx or dy

To decide whether to integrate with respect to x or y, look at the shape of the region.

- Use dx when vertical rectangles provide one simple top function and one simple bottom function.
$$A = \int_a^b (\text{top} - \text{bottom})\,dx$$
- Use dy when horizontal rectangles provide one simple right function and one simple

left function.

$$A = \int_c^d (\text{right} - \text{left})\, dy$$

Warning

When integrating with respect to y, the limits must be y-values, not x-values.
Also, use

$$\text{right} - \text{left},$$

not top minus bottom.

Example 24.5.

Find the area of the region bounded by

$$x = y^2 - 1 \quad \text{and} \quad x = \sqrt{y}.$$

Solution.

To find the points of intersection, set the two expressions for x equal:

$$y^2 - 1 = \sqrt{y}.$$

Since $\sqrt{y}$ requires $y \geq 0$, let

$$u = \sqrt{y}.$$

Then $y = u^2$, so

$$u^4 - u - 1 = 0.$$

This equation has one nonnegative solution,

$$u \approx 1.220744,$$

which gives

$$y = u^2 \approx 1.490216.$$

However, the two curves have only one point of intersection. Therefore, they do not enclose a bounded region by themselves.

$$\boxed{\text{The two curves alone do not enclose a bounded region.}}$$

If the intended region is also bounded by the x-axis, then the area is

$$A = \int_0^{1.490216} \left[\sqrt{y} - (y^2 - 1)\right] \, dy,$$

and

$$A = \left[\frac{2}{3}y^{3/2} - \frac{y^3}{3} + y\right]_0^{1.490216} \approx 1.59987.$$

Thus, with the x-axis included,

$$\boxed{A \approx 1.600}.$$

Example 24.6.

Find the area of the region bounded by

$$x = 2y - y^2$$

and the y-axis.

Solution.

The y-axis can also be written as

$$x = 0.$$

To find the points at where the curve intersects the y-axis:

$$2y - y^2 = 0.$$

By factoring,

$$y(2 - y) = 0.$$

Thus,

$$y = 0 \qquad \text{and} \qquad y = 2.$$

For $0 \le y \le 2$, the curve $x = 2y - y^2$ is to the right of the y-axis. Therefore, the area is

$$A = \int_0^2 \left[(2y - y^2) - 0\right] \, dy.$$

Hence,

$$A = \left[y^2 - \frac{y^3}{3}\right]_0^2.$$

$$A = 4 - \frac{8}{3} = \frac{4}{3}.$$

Thus, the area of the region is

$$\boxed{\frac{4}{3}}.$$

24.5. Summary and Strategy

When finding areas between curves, the most important step is to choose rectangles that simply describe the region.

Important

Summary of Area Formulas

Using vertical rectangles:

$$\boxed{A = \int_a^b (\text{top} - \text{bottom})\, dx}$$

Using horizontal rectangles:

$$\boxed{A = \int_c^d (\text{right} - \text{left})\, dy}$$

If the curves switch positions, split the region or use

$$\boxed{A = \int_a^b |f(x) - g(x)|\, dx}$$

Important

General Strategy

1. Sketch the curves.
2. Find intersection points.
3. Decide whether vertical or horizontal rectangles are easier.
4. Write the height/width of a typical rectangle.
5. Set up the integral.
6. Evaluate the integral.

Example 24.7.
Sketch the region enclosed by

$$y = (x-2)^2 \quad \text{and} \quad y = x.$$

Decide whether to integrate with respect to x or y, and set up the integral for the area.
Solution.
First, find the points of intersection:

$$(x-2)^2 = x.$$

By simplifying we get

$$x^2 - 5x + 4 = 0.$$

So,

$$(x-1)(x-4) = 0.$$

Therefore,

$$x = 1 \qquad \text{or} \qquad x = 4.$$

Th y-coordinates are

$$y = 1 \qquad \text{and} \qquad y = 4$$

respectively.

Thus, the points of intersection are

$$(1,1) \qquad \text{and} \qquad (4,4).$$

On the interval $1 \le x \le 4$, the line

$$y = x$$

lies above the parabola

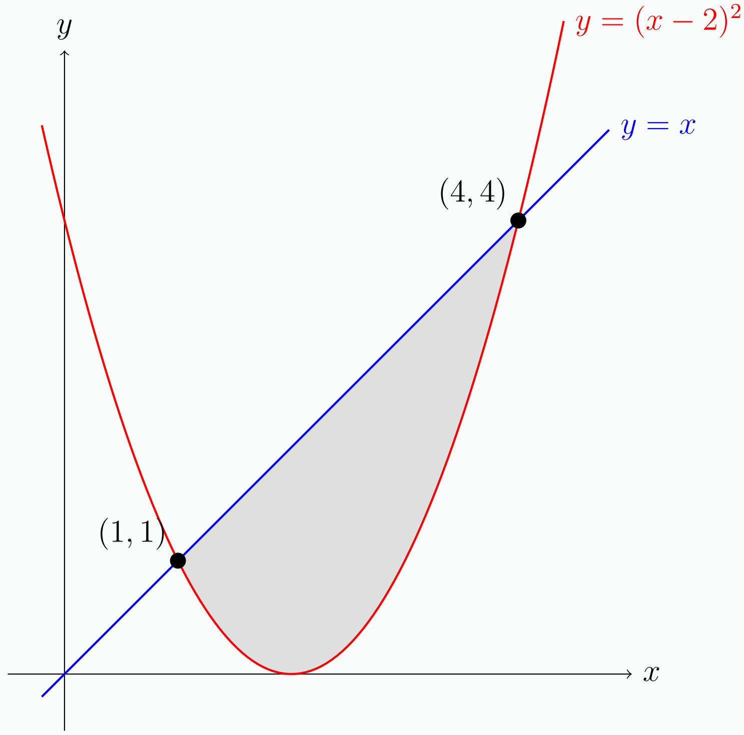
$$y = (x-2)^2.$$

Because both curves are already written as functions of x, it is simpler to integrate with respect to x.

Thus, the area is given by

$$\boxed{A = \int_1^4 \left[x - (x-2)^2\right]\, dx}.$$

A sketch of the region is given below.



Example 24.8.

Find the area of the region bounded by

$$y = \sqrt{x+3} \qquad \text{and} \qquad y = \frac{x+3}{2}.$$

Solution.

First, find the points of intersection:

$$\sqrt{x+3} = \frac{x+3}{2}.$$

Since $x + 3 \geq 0$, square both sides to get

$$x + 3 = \frac{(x+3)^2}{4}.$$

By simplifying we get

$$(x+3)\big((x+3)-4\big)=0.$$

Thus,

$$x=-3 \qquad \text{or} \qquad x=1.$$

The corresponding points of intersection are

$$(-3,0) \qquad \text{and} \qquad (1,2).$$

On the interval $[-3,1]$, the curve

$$y=\sqrt{x+3} \qquad \text{lies above the line} \qquad y=\frac{x+3}{2}.$$

Therefore, the area is

$$A=\int_{-3}^{1}\left(\sqrt{x+3}-\frac{x+3}{2}\right)dx.$$

By evaluating we get:

$$A=\left[\frac{2}{3}(x+3)^{3/2}-\frac{(x+3)^2}{4}\right]_{-3}^{1}.$$

$$A=\frac{2}{3}(4)^{3/2}-\frac{4^2}{4}=\frac{16}{3}-4=\frac{4}{3}.$$

Thus, the area of the region is

$$\boxed{\frac{4}{3}}.$$

Exercises

Find the area of the region bounded by the given curves.

1.
$$y = x + 1, \qquad y = 9 - x^2, \qquad x = -1, \qquad x = 2$$

2.
$$y = \sin x, \qquad y = x, \qquad x = \frac{\pi}{2}, \qquad x = \pi$$

3.
$$y = (x-2)^2, \qquad y = x$$

4.
$$y = x^2 - 4x, \qquad y = 2x$$

5.
$$y = \sqrt{x+3}, \qquad y = \frac{x+3}{2}$$

6.
$$y = 12 - x^2, \qquad y = x^2 - 6$$

7.
$$y = x^2, \qquad y = 4x - x^2$$

8.
$$y = \sec^2 x, \qquad y = 8\cos x, \qquad -\frac{\pi}{3} \leq x \leq \frac{\pi}{3}$$

9.
$$y = \cos x, \qquad y = 2 - \cos x, \qquad 0 \leq x \leq 2\pi$$

10.
$$x = 2y^2, \qquad x = 4 + y^2$$

11.
$$y = \sqrt{x-1}, \qquad x - y = 1$$

12.
$$y = \cos(\pi x), \qquad y = 4x^2 - 1$$

13.
$$x = y^4, \qquad y = \sqrt{2-x}, \qquad y = 0$$

14.
$$y = \cos x, \qquad y = 1 - \frac{2x}{\pi}$$

15. The graphs of two functions f and g enclose two regions. The area of the first region is 12, and the area of the second region is 27. Find the total area between the curves.

16. The graphs of two functions f and g enclose two regions. Suppose
$$\int_0^5 [f(x) - g(x)]\,dx = 15$$
and one of the regions has area 12. Find the area of the other region.

17. Evaluate the integral and interpret it as the area of a region:
$$\int_0^{\pi/2} |\sin x - \cos 2x|\,dx.$$

18. Evaluate the integral and interpret it as the area of a region:
$$\int_0^4 |\sqrt{x+2} - x|\,dx.$$

19. Find the area of the region bounded by the parabola
$$y = x^2,$$
the tangent line to this parabola at $(1, 1)$, and the x-axis.

20. Find the number b such that the line
$$y = b$$
divides the region bounded by
$$y = x^2 \qquad \text{and} \qquad y = 4$$
into two regions with equal area.

21. Find the number a such that the line $x = a$ bisects the area under the curve
$$y = \frac{1}{x^2}, \qquad 1 \le x \le 4.$$

Chapter 25

Volumes: Disk and Washer Methods

25.1. Volumes

Brief Description

In Chapter 23, we used definite integrals to find the area of a region. In this section, we use a similar idea to find the **volume** of a three-dimensional solid.

The basic strategy is simple:

1. Slice the solid into many thin pieces.
2. Find the volume of a typical slice.
3. Add the volumes of all slices using a definite integral.

Just as area is obtained by summing rectangles, volume will be obtained by summing thin slices of a solid.

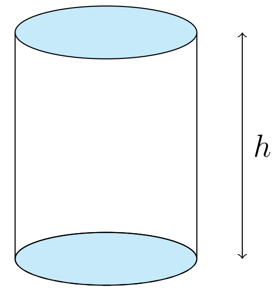
25.1.1 Volume of a Cylinder

Recall that the volume of a cylinder is the product of the area of its base and its height.

Volume of a cylinder

If a cylinder has base area A and height h, then its volume is

$$V = Ah.$$



Base Area A

25.1.2 Cross-Sections of a Solid

To find the volume of a more complicated solid, we imagine slicing it with planes perpendicular to the x-axis.

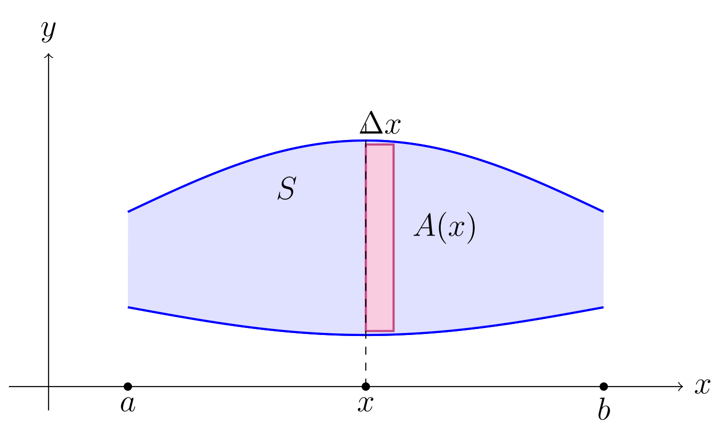
The intersection of the solid with one of these planes is called a **cross-section**.

Cross-Section

A **cross-section** of a solid is the plane region obtained by intersecting the solid with a plane.

If the cross-sectional area varies from slice to slice, we denote it by

$$A(x).$$



For each value of x, the slice has area $A(x)$.

As x changes, the cross-sectional area changes.

25.1.3 Approximating the Volume

Suppose the interval $[a, b]$ is divided into n subintervals of width

$$\Delta x = \frac{b-a}{n}.$$

Choose a sample point x_i^* in each subinterval.

The slice at x_i^* has cross-sectional area $A(x_i^*)$.

Its approximate volume is

$$A(x_i^*)\Delta x.$$

Adding the volumes of all slices gives

$$V \approx \sum_{i=1}^{n} A(x_i^*)\Delta x.$$

The exact volume is obtained by taking the limit as the number of slices becomes infinitely large.

Important

If $A(x)$ is the cross-sectional area of a solid at position x, then the volume of the solid is given by

$$V = \lim_{n\to\infty} \sum_{i=1}^{n} A(x_i^*)\Delta x = \int_a^b A(x)\,dx.$$

25.1.4 How We Will Use This Formula

Most of the solids we study will be **solids of revolution**.

These are solids obtained by rotating a plane region about a line.

For such solids:

- We first determine the cross-sectional area $A(x)$.
- Then we compute
$$V = \int_a^b A(x)\,dx.$$

In the next sections, we will learn efficient methods for finding $A(x)$, including:

- The Disk Method
- The Washer Method

Example 25.1.
A solid has cross-sectional area

$$A(x) = x^2 + 1, \qquad 0 \le x \le 2.$$

Find its volume.
Solution.

$$\begin{aligned} V &= \int_0^2 (x^2+1)\,dx \\ &= \left[\frac{x^3}{3} + x\right]_0^2 \\ &= \left(\frac{2^3}{3} + 2\right) - \left(\frac{0^3}{3} + 0\right) \\ &= \frac{8}{3} + 2. \end{aligned}$$

Thus, the volume of the solid is

$$\boxed{V = \frac{14}{3}.}$$

Example 25.2.
The cross-sections of a solid perpendicular to the x-axis have area

$$A(x) = 4 - x, \qquad 0 \le x \le 4.$$

Find the volume.
Solution.

$$\begin{aligned} V &= \int_0^4 (4-x)\,dx \\ &= \left[4x - \frac{x^2}{2}\right]_0^4 \\ &= \left(4\cdot 4 - \frac{4^2}{2}\right) - \left(4\cdot 0 - \frac{0^2}{2}\right) \\ &= 16 - \frac{16}{2} - 0. \end{aligned}$$

Thus, the volume of the solid is

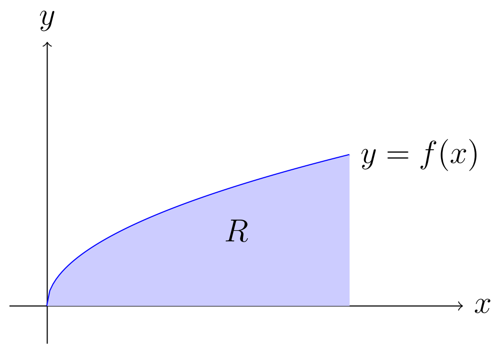
$$\boxed{V = 8.}$$

25.2. The Disk Method

Many solids are formed by rotating a plane region about a line.

Such solids are called **solids of revolution**.

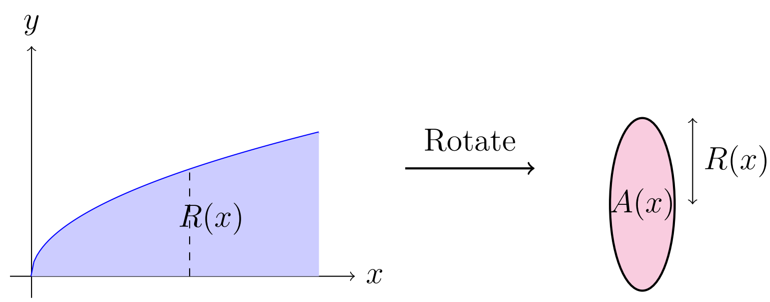
Suppose a region is revolved about an axis. If every cross-section perpendicular to the axis of rotation is a solid disk, then the volume can be found using the Disk Method.



When the region under the graph of $y = f(x)$ is revolved about the x-axis, a solid of revolution is formed.

A typical cross-section perpendicular to the x-axis is a disk whose radius is

$$R(x) = f(x).$$



The area of a cross-sectional disk is

$$A(x) = \pi[R(x)]^2.$$

Since

$$V = \int_a^b A(x)\,dx,$$

we obtain the Disk Method formula given below.

Important

Disk Method

If a region is revolved about an axis and the cross-sections perpendicular to the axis are disks of radius $R(x)$, then

$$V = \pi \int_a^b [R(x)]^2 \, dx.$$

Similarly, if the radius is expressed as a function of y, then

$$V = \pi \int_c^d [R(y)]^2 \, dy.$$

Example 25.3.

Find the volume of the solid obtained by revolving the region bounded by

$$y = \sqrt{x}, \qquad 0 \leq x \leq 4$$

about the x-axis.

Solution.

The radius of a typical disk is

$$R(x) = \sqrt{x}.$$

Using the Disk Method,

$$V = \pi \int_0^4 (\sqrt{x})^2 \, dx = \pi \int_0^4 x \, dx.$$

$$V = \pi \left[\frac{x^2}{2}\right]_0^4.$$

$$V = \pi(8).$$

$$\boxed{V = 8\pi}$$

Example 25.4.

Find the volume of the solid obtained by revolving the region bounded by

$$y = x^2, \qquad 0 \leq x \leq 2$$

about the x-axis.

Solution.
The radius is

$$R(x) = x^2.$$

Thus,

$$V = \pi \int_0^2 (x^2)^2 \, dx = \pi \int_0^2 x^4 \, dx.$$

$$V = \pi \left[\frac{x^5}{5} \right]_0^2.$$

$$V = \frac{32\pi}{5}.$$

$$\boxed{\frac{32\pi}{5}}$$

Example 25.5.
Find the volume of the solid obtained by revolving the region bounded by

$$x = y^2, \qquad 0 \leq y \leq 2$$

about the y-axis.
Solution.
The radius is

$$R(y) = y^2.$$

Using the Disk Method,

$$V = \pi \int_0^2 (y^2)^2 \, dy = \pi \int_0^2 y^4 \, dy.$$

$$V = \pi \left[\frac{y^5}{5} \right]_0^2.$$

$$V = \frac{32\pi}{5}.$$

$$\boxed{\frac{32\pi}{5}}$$

25.3. The Washer Method

Sometimes, rotating a region produces a solid with a hole in the middle.

In this case, the cross-sections perpendicular to the axis of rotation are not disks. Instead, they are **washers**.

A washer is like a disk with a smaller disk removed from the center.

$$\text{Area of washer} = \pi(\text{outer radius})^2 - \pi(\text{inner radius})^2.$$

Therefore,

$$A(x) = \pi[R(x)]^2 - \pi[r(x)]^2,$$

where $R(x)$ is the outer radius and $r(x)$ is the inner radius.

Important

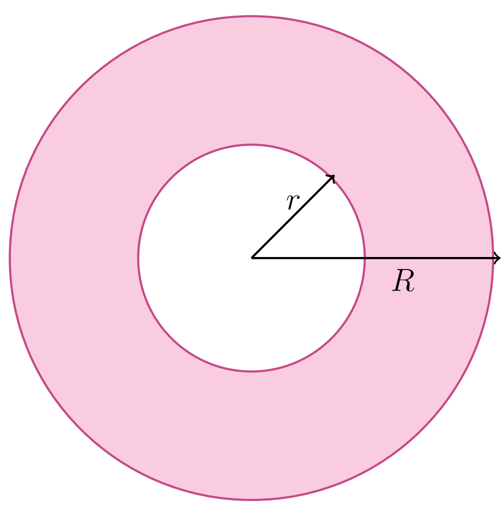
Washer Method

If a region is revolved about an axis and the cross-sections perpendicular to the axis are washers, then

$$V = \pi \int_a^b \left([R(x)]^2 - [r(x)]^2\right)\, dx.$$

Similarly, if the radii are functions of y, then

$$V = \pi \int_c^d \left([R(y)]^2 - [r(y)]^2\right)\, dy.$$



$$A = \pi R^2 - \pi r^2$$

25.3.1 Washer Method About the x-Axis

Suppose a region lies between two curves

$$y = f(x) \qquad \text{and} \qquad y = g(x),$$

where

$$f(x) \geq g(x) \geq 0.$$

If this region is revolved about the x-axis, then the outer radius is

$$R(x) = f(x),$$

and the inner radius is

$$r(x) = g(x).$$

Thus,

$$V = \pi \int_a^b \left([f(x)]^2 - [g(x)]^2\right)\, dx.$$

Example 25.6.
Find the volume of the solid obtained by revolving the region bounded by

$$y = x, \qquad y = x^2, \qquad 0 \leq x \leq 1$$

about the x-axis.

Solution.
On $0 \leq x \leq 1$, we have

$$x \geq x^2.$$

So the outer radius is

$$R(x) = x,$$

and the inner radius is

$$r(x) = x^2.$$

Using the Washer Method,

$$V = \pi \int_0^1 \left(x^2 - (x^2)^2\right)\, dx.$$

$$V = \pi \int_0^1 (x^2 - x^4)\,dx.$$

$$V = \pi \left[\frac{x^3}{3} - \frac{x^5}{5}\right]_0^1.$$

$$V = \pi \left(\frac{1}{3} - \frac{1}{5}\right).$$

$$V = \frac{2\pi}{15}.$$

$$\boxed{V = \frac{2\pi}{15}}$$

25.3.2 Washer Method About a Horizontal Line

If the axis of rotation is not the x-axis, the radius is found by measuring vertical distance from the curve to the axis of rotation.

Important

When rotating about a horizontal line, radii are vertical distances.

$$\text{radius} = \text{top value} - \text{bottom value}.$$

Example 25.7.

Find the volume of the solid obtained by revolving the region bounded by

$$y = x^2, \qquad y = 1, \qquad 0 \le x \le 1$$

about the line

$$y = 1.$$

Solution.

The axis of rotation is $y = 1$.
The region lies below $y = 1$, so the radius is the distance from $y = 1$ to $y = x^2$:

$$R(x) = 1 - x^2.$$

There is no hole, so this is actually a disk problem:

$$r(x) = 0.$$

Thus,

$$V = \pi \int_0^1 (1 - x^2)^2 \, dx.$$

Expand:

$$(1 - x^2)^2 = 1 - 2x^2 + x^4.$$

So

$$V = \pi \int_0^1 (1 - 2x^2 + x^4) \, dx.$$

$$V = \pi \left[x - \frac{2x^3}{3} + \frac{x^5}{5} \right]_0^1.$$

$$V = \pi \left(1 - \frac{2}{3} + \frac{1}{5} \right).$$

$$V = \pi \left(\frac{8}{15} \right).$$

$$\boxed{V = \frac{8\pi}{15}}$$

Example 25.8.
Find the volume of the solid obtained by revolving the region bounded by

$$y = x, \qquad y = x^2, \qquad 0 \le x \le 1$$

about the line

$$y = 2.$$

Solution.
The axis of rotation is $y = 2$, which is above the region.
The outer radius is the distance from $y = 2$ to the lower curve $y = x^2$:

$$R(x) = 2 - x^2.$$

The inner radius is the distance from $y = 2$ to the upper curve $y = x$:

$$r(x) = 2 - x.$$

Using the Washer Method,

$$V = \pi \int_0^1 \left[(2 - x^2)^2 - (2 - x)^2\right] \, dx.$$

Expand:

$$(2 - x^2)^2 = 4 - 4x^2 + x^4,$$

and

$$(2 - x)^2 = 4 - 4x + x^2.$$

Therefore,

$$(2 - x^2)^2 - (2 - x)^2 = 4x - 5x^2 + x^4.$$

So

$$V = \pi \int_0^1 (4x - 5x^2 + x^4) \, dx.$$

$$V = \pi \left[2x^2 - \frac{5x^3}{3} + \frac{x^5}{5}\right]_0^1.$$

$$V = \pi \left(2 - \frac{5}{3} + \frac{1}{5}\right).$$

$$V = \pi \left(\frac{8}{15}\right).$$

$$\boxed{V = \frac{8\pi}{15}}$$

25.3.3 Washer Method About the y-Axis

When a region is revolved about the y-axis, it is often better to write the curves as functions of y. In that case, the formula becomes

$$V = \pi \int_c^d \left([R(y)]^2 - [r(y)]^2\right)\, dy.$$

Example 25.9.

Find the volume of the solid obtained by revolving the region bounded by

$$x = 2y - y^2$$

and the y-axis about the y-axis.

Solution.

The y-axis is

$$x = 0.$$

The curve is

$$x = 2y - y^2.$$

Find the y-values where the curve meets the y-axis:

$$2y - y^2 = 0.$$

$$y(2 - y) = 0.$$

Thus,

$$y = 0 \qquad \text{or} \qquad y = 2.$$

The radius is

$$R(y) = 2y - y^2.$$

There is no hole, so

$$r(y) = 0.$$

Therefore,

$$V = \pi \int_0^2 (2y - y^2)^2 \, dy.$$

Expand:

$$(2y - y^2)^2 = 4y^2 - 4y^3 + y^4.$$

So

$$V = \pi \int_0^2 (4y^2 - 4y^3 + y^4) \, dy.$$

$$V = \pi \left[\frac{4y^3}{3} - y^4 + \frac{y^5}{5} \right]_0^2.$$

$$V = \pi \left(\frac{32}{3} - 16 + \frac{32}{5} \right).$$

$$V = \pi \left(\frac{16}{15} \right).$$

$$\boxed{V = \frac{16\pi}{15}}$$

Warning

For washers, always identify the radii carefully.

$$R = \text{outer radius}$$

$$r = \text{inner radius}$$

Then use

$$V = \pi \int (R^2 - r^2).$$

Do not subtract radii first. In general,

$$R^2 - r^2 \neq (R - r)^2.$$

Exercises

Basic Volume Problems

Find the volume of each solid.

1. Cross-sectional area:
$$A(x) = 3x + 2, \qquad 0 \le x \le 5.$$
2. Cross-sectional area:
$$A(x) = x^2, \qquad 0 \le x \le 3.$$
3. Cross-sectional area:
$$A(x) = 9 - x^2, \qquad -3 \le x \le 3.$$
4. Cross-sectional area:
$$A(x) = \sqrt{x}, \qquad 0 \le x \le 4.$$
5. Cross-sectional area:
$$A(x) = e^x, \qquad 0 \le x \le 1.$$

The Disk Method

Find the volume of the solid using the Disk Method.

1. Revolve the region bounded by
$$y = x + 1, \qquad y = 0, \qquad x = 0, \qquad x = 2$$
about the x-axis.
2. Revolve the region bounded by
$$y = \frac{1}{x}, \qquad y = 0, \qquad x = 1, \qquad x = 4$$
about the x-axis.
3. Revolve the region bounded by
$$y = \sqrt{x - 1}, \qquad y = 0, \qquad x = 5$$
about the x-axis.

4. Revolve the region bounded by

$$x = 2\sqrt{y}, \qquad x = 0, \qquad y = 9$$

about the y-axis.

5. Revolve the region bounded by

$$2x = y^2, \qquad x = 0, \qquad y = 4$$

about the y-axis.

The Washer Method

Find the volume of the solid obtained by rotating the region about the specified line.

1. $$y = x^3, \qquad y = x, \qquad x \geq 0;$$

   about the x-axis.

2. $$y = 6 - x^2, \qquad y = 2;$$

   about the x-axis.

3. $$y = x^2, \qquad x = y^2;$$

   about $y = 1$.

4. $$y = x^3, \qquad y = 1, \qquad x = 2;$$

   about $y = -3$.

5. $$y = \sin x, \qquad y = \cos x, \qquad 0 \leq x \leq \frac{\pi}{4};$$

   about $y = -1$.

6. $$y = x^3, \qquad y = 0, \qquad x = 1;$$

   about $x = 2$.

7.
$$x = y^2, \qquad x = 1 - y^2;$$
about $x = 3$.

8.
$$y = \tan x, \qquad y = 0, \qquad x = \frac{\pi}{4};$$
about $y = -1$.

9.
$$y = 0, \qquad y = \cos^2 x, \qquad -\frac{\pi}{2} \leq x \leq \frac{\pi}{2};$$
about $y = 1$.

Chapter 26

Volumes: Cylindrical Shell Method

26.1. Volumes by Cylindrical Shells

Brief Description

In Chapter 25, we computed volumes of solids of revolution using the **Disk Method** and the **Washer Method**. However, these methods are not always the most convenient. Sometimes revolving a region about an axis produces washers whose radii are difficult to describe algebraically. In such cases, it is often easier to use the **Method of Cylindrical Shells**.

The idea is simple:

- Instead of slicing the region *perpendicular* to the axis of rotation, we slice it *parallel* to the axis of rotation.
- When a thin rectangle is revolved about the axis, it sweeps out a thin cylindrical shell.
- We add the volumes of all such shells using integration.

Important

For the Shell Method, each shell contributes

$$\text{Volume} = \text{Circumference} \times \text{Height} \times \text{Thickness}.$$

Since

$$\text{Circumference} = 2\pi \times \text{Radius},$$

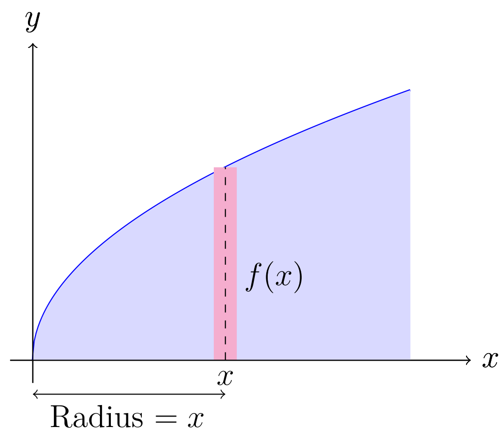
the shell volume is approximately

$$dV = 2\pi \times \text{Radius} \times \text{Height} \times \text{Thickness}.$$

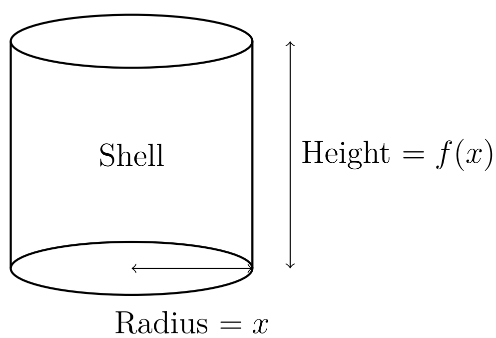
26.1.1 A Typical Cylindrical Shell

Suppose the region under the curve $y = f(x)$ is **revolved about the y-axis.**

A vertical rectangle at position x produces a cylindrical shell.



When this rectangle is revolved about the y-axis, it produces a cylindrical shell.



Therefore,

$$\boxed{dV = 2\pi x\, f(x)\, dx.}$$

26.1.2 The Shell Formula About the y-Axis

Important

Suppose a region bounded by

$$y = f(x), \qquad y = 0, \qquad a \le x \le b$$

is revolved about the y-axis.

Then the volume is

$$\boxed{V = 2\pi \int_a^b (\text{Radius} \times \text{Height})\, dx}$$

or equivalently

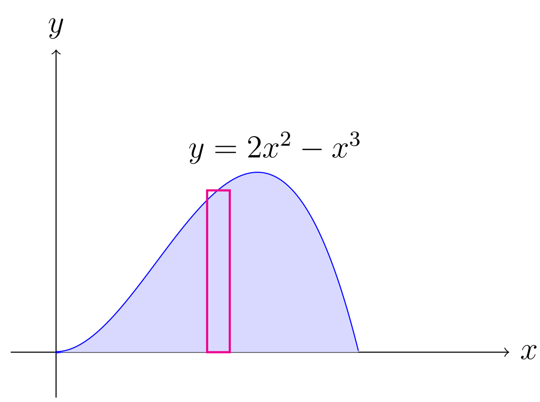
$$\boxed{V = 2\pi \int_a^b x\, f(x)\, dx.}$$

26.1.3 Why Use Cylindrical Shells?

Sometimes, the Washer Method requires solving a complicated equation for x in terms of y. For example, suppose we rotate the region under

$$y = 2x^2 - x^3, \qquad 0 \le x \le 2,$$

about the y-axis.



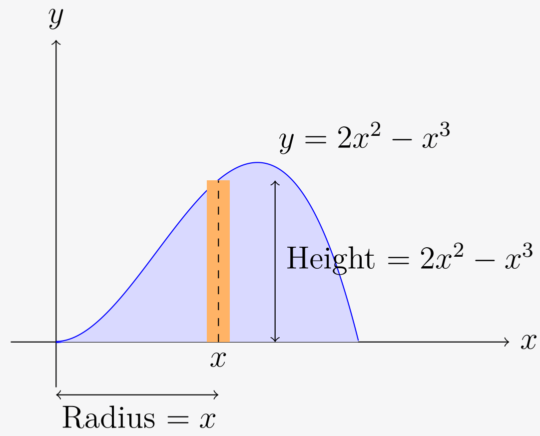
At first glance, both the Washer Method and the Shell Method appear to be possible. Let us compare them.

Important

Using Cylindrical Shells

A vertical slice produces a cylindrical shell.
The dimensions of the shell are immediately available:

$$\text{Radius} = x, \qquad \text{Height} = 2x^2 - x^3.$$



Therefore,

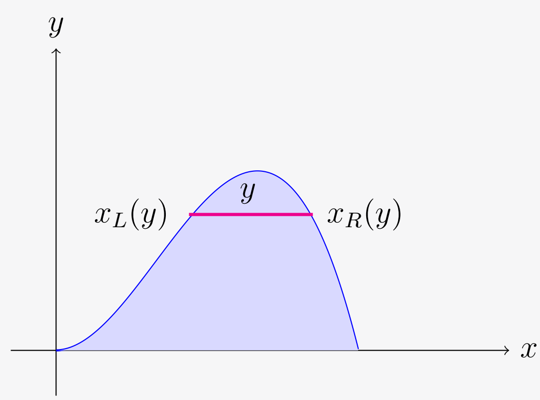
$$V = 2\pi \int_0^2 x(2x^2 - x^3)\,dx.$$

The integral can be set up directly from the given function.

Important

Using Washers

To use washers, we must slice perpendicular to the axis of rotation. Since the axis of rotation is the y-axis, the slices must be horizontal.



A typical washer has

$$\text{Outer Radius} : r_{\text{out}} = x_R(y), \qquad \text{Inner Radius} : r_{\text{in}} = x_L(y).$$

Thus,

$$V = \pi \int \left(x_R(y)^2 - x_L(y)^2 \right) dy.$$

The problem is that neither radius is known explicitly.

To find $x_L(y)$ and $x_R(y)$, we must solve

$$y = 2x^2 - x^3$$

for x.
Rearranging gives

$$x^3 - 2x^2 + y = 0.$$

This is a cubic equation.
Unlike a quadratic equation, there is no simple formula that gives a convenient expression for x as a function of y. Furthermore, most horizontal lines intersect the curve at two points, producing two different inverse branches:

$$x_L(y) \qquad \text{and} \qquad x_R(y).$$

Consequently, the washer method would require finding these two functions before the volume integral can even be set up.

Important

Conclusion
For this problem, the Shell Method is clearly the better choice.
The shell dimensions are obtained directly from the graph:

$$\text{radius} = x, \qquad \text{height} = 2x^2 - x^3.$$

As a result,

$$V = 2\pi \int_0^2 x(2x^2 - x^3)\, dx$$

can be written immediately.
The Washer Method, on the other hand, requires solving the cubic equation

$$x^3 - 2x^2 + y = 0$$

to obtain $x_L(y)$ and $x_R(y)$, making the setup much more difficult.

Example 26.1.
Find the volume of the solid obtained by rotating the region bounded by

$$y = x, \qquad y = 0, \qquad 0 \le x \le 2$$

about the y-axis.

Solution.
Using cylindrical shells,

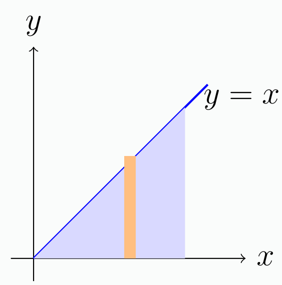
$$\text{radius} = x, \qquad \text{height} = x.$$

Therefore,

$$V = 2\pi \int_0^2 x(x)\,dx.$$

$$= 2\pi \int_0^2 x^2\,dx = 2\pi \left[\frac{x^3}{3}\right]_0^2.$$

$$= 2\pi\left(\frac{8}{3}\right) = \boxed{\frac{16\pi}{3}}.$$



Warning

For the Shell Method:

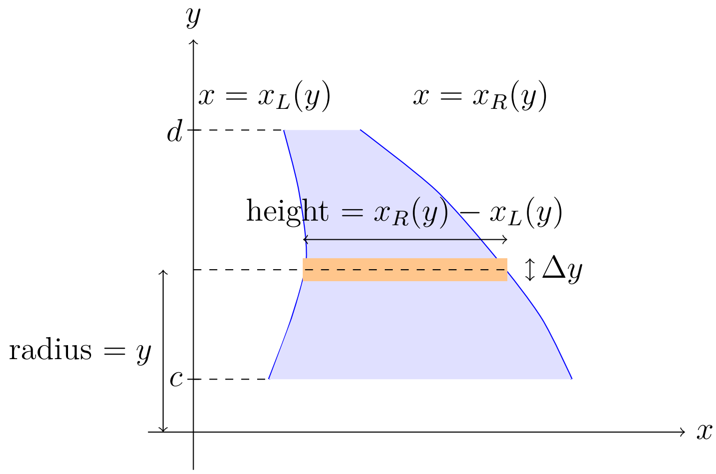
$$\boxed{V = 2\pi \int (\text{radius})(\text{height})\,(\text{thickness})}$$

Students often forget to include the radius factor.
Without the radius, the formula is incorrect.

26.1.4 The Shell Method About the x-Axis

The Shell Method can also be used **when the axis of rotation is the x-axis.**

In this case, we use **horizontal rectangles**. When a horizontal rectangle is revolved about the x-axis, it produces a cylindrical shell.



A typical shell has

$$\text{Radius} = y, \qquad \text{Height} = x_R(y) - x_L(y),$$

and thickness

$$\Delta y.$$

Therefore,

$$\Delta V \approx 2\pi y\big(x_R(y) - x_L(y)\big)\Delta y.$$

Taking the limit gives

$$V = 2\pi \int_c^d y\big(x_R(y) - x_L(y)\big)\, dy.$$

Important

Shell Method About the x-Axis

If a horizontal slice produces a cylindrical shell, then

$$\boxed{V = 2\pi \int_c^d (\text{Radius} \times \text{Height})\, dy}$$

where the radius is measured from the shell to the axis of rotation.

26.1.5 Rotation About the x-Axis

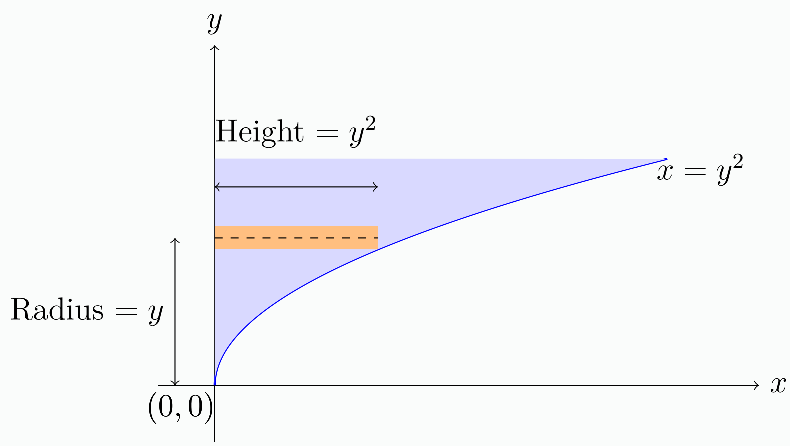
Example 26.2.

Find the volume of the solid obtained by revolving the region bounded by

$$x = y^2, \qquad 0 \le y \le 2$$

about the x-axis.

Solution.



Using horizontal shells,

$$\text{Radius} = y, \qquad \text{Height} = y^2.$$

Therefore,

$$V = 2\pi \int_0^2 y(y^2)\,dy.$$

$$= 2\pi \int_0^2 y^3\,dy.$$

$$= 2\pi \left[\frac{y^4}{4}\right]_0^2.$$

$$= 2\pi(4).$$

$$\boxed{V = 8\pi}$$

26.2. Finding Radius and Height

The most important step in the Shell Method is correctly identifying the radius and height.

Important

For every shell problem:

$$V = 2\pi \int (\text{Radius} \times \text{Height} \times \text{ Thickness})$$

Ask yourself:

1. What is the distance from the shell to the axis of rotation? (This is the radius.)
2. What is the length of the shell? (This is the height.)
3. Is the thickness dx or dy?

Example 26.3.

Find the volume of the solid obtained by revolving the region bounded by

$$y = x^2, \qquad y = 0, \qquad x = 2$$

about the y-axis.

Solution.

A vertical slice produces a cylindrical shell.

$$\text{radius} = x \qquad \text{and} \qquad \text{height} = x^2.$$

Therefore,

$$V = 2\pi \int_0^2 x(x^2)\,dx.$$

$$= 2\pi \int_0^2 x^3\,dx.$$

$$= 2\pi \left[\frac{x^4}{4}\right]_0^2.$$

$$= 2\pi(4).$$

$$\boxed{V = 8\pi}$$

Example 26.4.
Find the volume of the solid obtained by revolving the region bounded by

$$y = \sqrt{x}, \qquad y = 0, \qquad x = 4$$

about the y-axis.
Solution.
Using cylindrical shells,

$$\text{radius} = x,$$

and

$$\text{height} = \sqrt{x}.$$

Hence

$$V = 2\pi \int_0^4 x\sqrt{x}\, dx.$$

$$= 2\pi \int_0^4 x^{3/2}\, dx.$$

$$= 2\pi \left[\frac{2}{5}x^{5/2}\right]_0^4.$$

$$= 2\pi \left(\frac{2}{5} \cdot 32\right).$$

$$= \frac{128\pi}{5}.$$

$$\boxed{\frac{128\pi}{5}}$$

Example 26.5.
Find the volume of the solid obtained by revolving the region bounded by

$$x = y^2, \qquad x = 4, \qquad y \geq 0$$

about the x-axis.
Solution.

The curves intersect when

$$y^2 = 4,$$

so

$$y = 2.$$

Using horizontal shells,

$$\text{radius} = y \qquad \text{and} \qquad \text{height} = 4 - y^2.$$

Thus,

$$V = 2\pi \int_0^2 y(4 - y^2)\, dy.$$
$$= 2\pi \int_0^2 (4y - y^3)\, dy.$$
$$= 2\pi \left[2y^2 - \frac{y^4}{4}\right]_0^2.$$
$$= 2\pi(8 - 4).$$
$$= 8\pi.$$
$$\boxed{V = 8\pi}$$

Warning

Students often confuse the radius and height.
Remember:

$$\text{radius} = \text{distance to the axis of rotation},$$

while

$$\text{height} = \text{length of the rectangle}.$$

Do not interchange them.

26.3. Choosing Between Washers and Shells

For many solids of revolution, both the Washer Method and the Shell Method can be used. The best method is usually the one that produces a simpler integral.

Important

Choosing a Method

Method	Slices
Washers	Perpendicular to the axis of rotation
Shells	Parallel to the axis of rotation

A good strategy is:

1. Sketch the region.
2. Decide whether vertical or horizontal slices are easier.
3. Determine whether those slices produce washers or shells.
4. Choose the method with the simpler integral.

Example 26.6.

Find the volume generated by revolving the region bounded by

$$y = \sqrt{x}$$

and

$$y = x^2$$

about the y-axis.

Solution.

First find the points of intersection:

$$\sqrt{x} = x^2.$$

Since $x \geq 0$, squaring both sides gives

$$x = x^4.$$

Thus

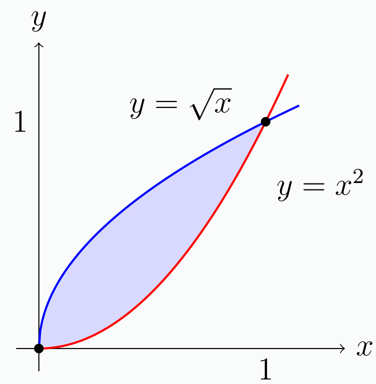
$$x(x^3 - 1) = 0,$$

so

$$x = 0 \qquad \text{or} \qquad x = 1.$$

Therefore, the region lies on

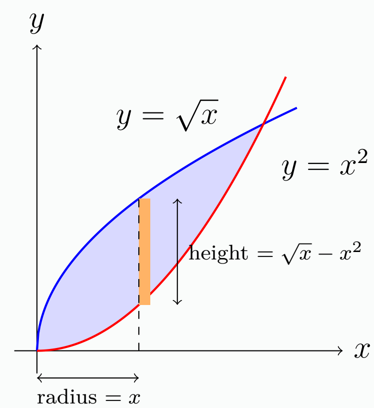
$$0 \le x \le 1.$$



Method 1: Shell Method

Since the region is revolved about the y-axis, vertical rectangles produce cylindrical shells. A typical vertical shell has

$$\text{radius} = x \qquad \text{and} \qquad \text{height} = \sqrt{x} - x^2.$$



Using cylindrical shells,

$$V = 2\pi \int_0^1 x(\sqrt{x} - x^2)\, dx.$$

$$= 2\pi \int_0^1 \left(x^{3/2} - x^3\right)\, dx.$$

$$= 2\pi \left[\frac{2}{5}x^{5/2} - \frac{x^4}{4}\right]_0^1.$$

$$= 2\pi\left(\frac{2}{5} - \frac{1}{4}\right).$$

$$= 2\pi\left(\frac{8-5}{20}\right) = 2\pi\left(\frac{3}{20}\right).$$

$$\boxed{V = \frac{3\pi}{10}}$$

Method 2: Washer Method

Since the axis of rotation is the y-axis, washers require horizontal slices. Therefore, we must solve each curve for x in terms of y.

From

$$y = \sqrt{x},$$

we get

$$x = y^2.$$

From

$$y = x^2,$$

we get

$$x = \sqrt{y}.$$

The region lies between

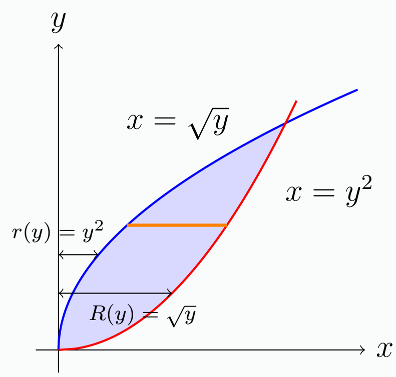
$$0 \le y \le 1.$$

For a horizontal washer,

$$R(y) = \sqrt{y},$$

and

$$r(y) = y^2.$$



Using the Washer Method,

$$V = \pi \int_0^1 \left[(\sqrt{y})^2 - (y^2)^2 \right] \, dy.$$

$$= \pi \int_0^1 (y - y^4) \, dy.$$

$$= \pi \left[\frac{y^2}{2} - \frac{y^5}{5} \right]_0^1 .$$

$$= \pi \left(\frac{1}{2} - \frac{1}{5} \right).$$

$$= \pi \left(\frac{5-2}{10} \right).$$

$$\boxed{V = \frac{3\pi}{10}}$$

Both methods give the same volume:

$$\boxed{V = \frac{3\pi}{10}.}$$

For this example, the Shell Method is slightly easier because it uses the original functions directly:

$$y = \sqrt{x} \qquad \text{and} \qquad y = x^2.$$

The Washer Method requires rewriting both curves as functions of y.

Example 26.7.

Find the volume generated by revolving the region bounded by

$$y = x - x^2$$

and the x-axis about the y-axis.

Solution.

The curve intersects the x-axis when

$$x - x^2 = 0.$$

$$x(1-x) = 0.$$

Therefore,

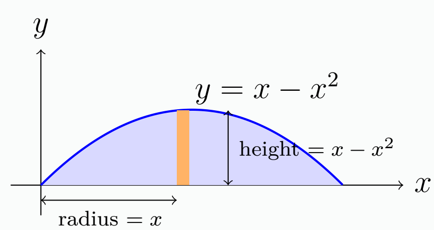
$$x = 0 \qquad \text{or} \qquad x = 1.$$

So the region lies on

$$0 \le x \le 1.$$

Since the region is revolved about the y-axis, vertical rectangles produce cylindrical shells.

$$\text{radius} = x \qquad \text{and} \qquad \text{height} = x - x^2.$$



Using the Shell Method,

$$V = 2\pi \int_0^1 x(x - x^2)\,dx.$$

$$= 2\pi \int_0^1 (x^2 - x^3)\,dx.$$

$$= 2\pi \left[\frac{x^3}{3} - \frac{x^4}{4}\right]_0^1.$$

$$= 2\pi \left(\frac{1}{3} - \frac{1}{4}\right).$$

$$= 2\pi \left(\frac{1}{12} \right).$$

$$\boxed{V = \frac{\pi}{6}}$$

Warning

When comparing methods, do not automatically choose washers.
Ask:

- Will I need to solve for x in terms of y?
- Will I need to solve for y in terms of x?
- Will I need to split the integral?
- Which method produces the simpler integrand?

Many volume problems are designed so that the Shell Method avoids difficult algebra.

26.3.1 More Examples Involving Cylindrical Shells

In many applications, the axis of rotation is not one of the coordinate axes. The Shell Method still works; the only difference is that the radius must be measured from the shell to the axis of rotation.

Important

Radius Formula
The radius is always the distance from the shell to the axis of rotation.

Axis of Rotation	Radius
y-axis ($x = 0$)	x
x-axis ($y = 0$)	y
$x = a$	$\lvert x - a \rvert$
$y = b$	$\lvert y - b \rvert$

Example 26.8.

Find the volume generated when the region bounded by

$$y = x - x^2$$

and the x-axis is revolved about the line

$$x = 2.$$

Solution.

The curve intersects the x-axis when

$$x - x^2 = 0,$$

so

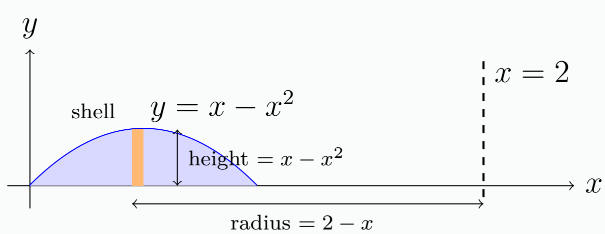
$$x = 0 \qquad \text{and} \qquad x = 1.$$

Using vertical shells,

$$\text{radius} = 2 - x,$$

and

$$\text{height} = x - x^2.$$



Therefore,

$$V = 2\pi \int_0^1 (2 - x)(x - x^2)\, dx.$$

Expanding,

$$(2 - x)(x - x^2) = 2x - 3x^2 + x^3.$$

Hence,

$$V = 2\pi \int_0^1 (2x - 3x^2 + x^3)\, dx.$$

$$= 2\pi \left[x^2 - x^3 + \frac{x^4}{4}\right]_0^1.$$

$$= 2\pi \left(1 - 1 + \frac{1}{4}\right).$$

$$= \boxed{\frac{\pi}{2}}.$$

Example 26.9.

Find the volume generated when the region bounded by

$$x = y^2, \qquad x = 4, \qquad y \geq 0$$

is revolved about the line

$$y = -1.$$

Solution.

First find the limits of integration.

$$y^2 = 4$$

gives

$$y = 2.$$

Thus

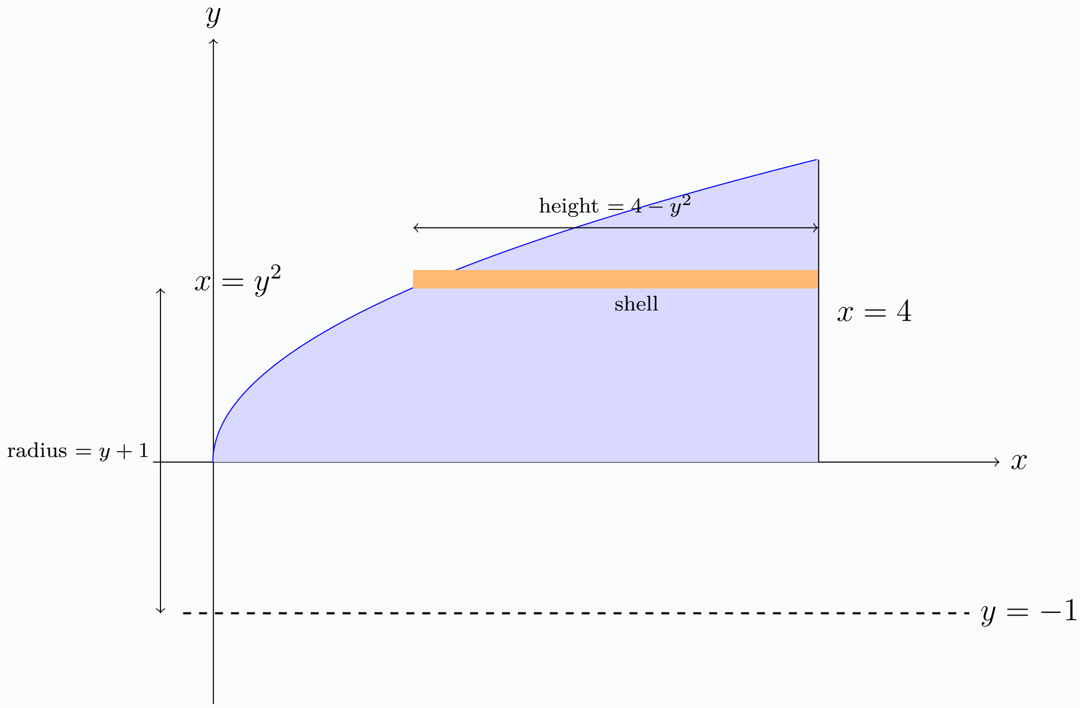
$$0 \leq y \leq 2.$$

Using horizontal shells,

$$\text{radius} = y + 1,$$

and

$$\text{height} = 4 - y^2.$$



Therefore,

$$V = 2\pi \int_0^2 (y+1)(4-y^2)\,dy.$$

Expanding,

$$(y+1)(4-y^2) = 4 + 4y - y^2 - y^3.$$

Thus,

$$V = 2\pi \int_0^2 (4 + 4y - y^2 - y^3)\,dy.$$

$$= 2\pi \left[4y + 2y^2 - \frac{y^3}{3} - \frac{y^4}{4}\right]_0^2.$$

$$= 2\pi \left(8 + 8 - \frac{8}{3} - 4\right).$$

$$= 2\pi \left(\frac{28}{3}\right).$$

$$= \boxed{\frac{56\pi}{3}}.$$

Example 26.10.

Find the volume generated when the region bounded by

$$y = x^2, \qquad y = 4$$

is revolved about the y-axis.

Solution.

The curves intersect when

$$x^2 = 4.$$

Since the region is symmetric about the y-axis, we may use shells on the interval

$$0 \leq x \leq 2.$$

A typical shell has

$$\text{radius} = x,$$

and

$$\text{height} = 4 - x^2.$$

Thus,

$$V = 2\pi \int_0^2 x(4 - x^2)\, dx.$$

$$= 2\pi \int_0^2 (4x - x^3)\, dx.$$

$$= 2\pi \left[2x^2 - \frac{x^4}{4}\right]_0^2.$$

$$= 2\pi(8 - 4).$$

$$= \boxed{8\pi}.$$

Warning

Common Mistakes

1. Using the wrong radius.

   The radius is the distance from the shell to the axis of rotation.

2. Using the wrong height.

   The height is the length of the rectangle.

3. Forgetting the factor 2π.

   The Shell Method formula is

$$V = 2\pi \int (\text{radius})(\text{height})\,(\text{thickness}).$$

4. Using washers when shells are easier.

   Always examine both methods before beginning the problem.

Exercises

Part A

Use the Shell Method to find the volume.

1. $$y = x^3, \qquad 0 \le x \le 1,$$

   rotated about the y-axis.

2. $$y = x^2, \qquad 0 \le x \le 3,$$

   rotated about the y-axis.

3. $$y = \sqrt{x}, \qquad 0 \le x \le 9,$$

   rotated about the y-axis.

4. $$x = y^2, \qquad 0 \le y \le 3,$$

   rotated about the x-axis.

5. $$x = 9 - y^2, \qquad 0 \le y \le 3,$$

   rotated about the x-axis.

6. $$y = x - x^2,$$

   rotated about the y-axis.

7. $$y = 4 - x^2, \qquad y = 0,$$

   rotated about the y-axis.

8. $$x = y^2, \qquad x = 16,$$

   rotated about the x-axis.

Part B

Use the Method of Cylindrical Shells to find the volume.

1. Revolve the region bounded by

$$y = x^2, \qquad y = 0, \qquad x = 3$$

   about the y-axis.

2. Revolve the region bounded by

$$y = \sqrt{x}, \qquad y = 0, \qquad x = 9$$

   about the y-axis.

3. Revolve the region bounded by

$$y = x - x^2$$

   and the x-axis about the line $x = 2$.

4. Revolve the region bounded by

$$y = 4 - x^2, \qquad y = 0$$

   about the y-axis.

5. Revolve the region bounded by

$$x = y^2, \qquad x = 9$$

   about the line $y = -2$.

6. Revolve the region bounded by

$$y = x^3, \qquad y = 8$$

   about the y-axis.

7. Revolve the region bounded by

$$x = 4 - y^2, \qquad x = 0$$

   about the line $y = -1$.

8. Revolve the region bounded by

$$y = \sin x, \qquad y = 0, \qquad 0 \le x \le \pi$$

about the y-axis.

9. Revolve the region bounded by

$$y = \cos x, \qquad y = 0, \qquad 0 \le x \le \frac{\pi}{2}$$

about the y-axis.

10. Revolve the region bounded by

$$x = y^3, \qquad x = 8$$

about the line $y = -1$.

Chapter 27

Work

27.1. Work

27.1.1 Work Done by a Variable Force

In everyday language, the word *work* means effort. In physics, however, work has a precise mathematical meaning.

Suppose a force F acts on an object and moves it a distance d in the direction of the force. If the force is constant, then the work done is

$$W = Fd.$$

For example, if a force of 10 N pushes an object 5 m, then

$$W = (10)(5) = 50 \text{ J}.$$

The unit of work is the **joule (J)**:

$$1 \text{ J} = 1 \text{ N} \cdot \text{m}.$$

In many real world situations, the force is *not constant*. For example,

- stretching a spring,
- lifting a rope,
- pumping water from a tank,
- moving an object through a varying force field.

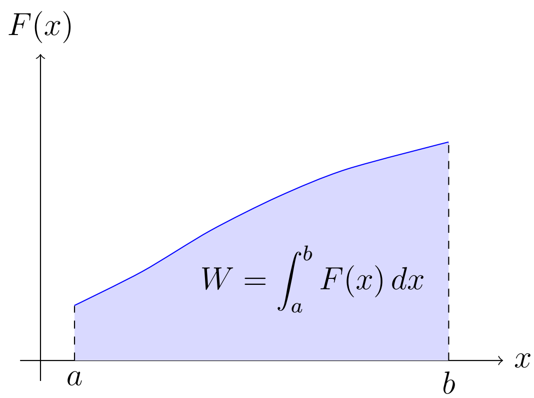
In these situations, we must use integration.

Work Done by a Variable Force

Suppose a variable force $F(x)$ moves an object along the x-axis from $x = a$ to $x = b$. Then the work done is

$$W = \int_a^b F(x)\,dx.$$

The units of W are joules (J) in the SI system.



The work done by a variable force is the area under the force curve.

Important

If $F(x) \geq 0$ on $[a, b]$, then the work done is equal to the area under the graph of $F(x)$ from a to b.

27.1.2 Derivation of the Work Formula

Suppose a force $F(x)$ varies as an object moves from $x = a$ to $x = b$.

Divide the interval into n small subintervals of width

$$\Delta x.$$

On a small interval $[x_{i-1}, x_i]$, the force is approximately constant and can be approximated by

$$F(x_i^*),$$

where x_i^* is a sample point in the interval.

Thus the work done on this small piece is approximately

$$\Delta W_i \approx F(x_i^*)\Delta x.$$

Adding all pieces gives

$$W \approx \sum_{i=1}^{n} F(x_i^*)\Delta x.$$

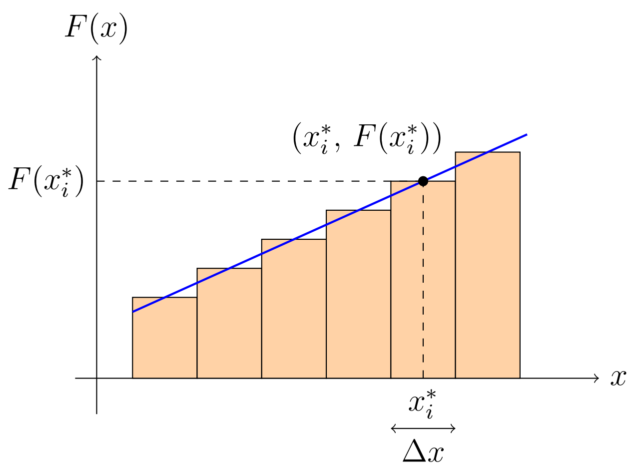
Taking the limit as $n \to \infty$,

$$W = \lim_{n\to\infty} \sum_{i=1}^{n} F(x_i^*)\Delta x.$$

Since this is a Riemann sum, we obtain

$$\boxed{W = \int_a^b F(x)\,dx}$$

which is the fundamental formula for work.



Important

The work formula

$$W = \int_a^b F(x)\,dx$$

is simply the limit of a Riemann sum where

$$\text{Work} = \text{Force} \times \text{Distance}.$$

27.1.3 Example: Constant Force

Example 27.1.

A constant force of 15 N moves an object 8 m.
Find the work done.

Solution.

Since the force is constant,

$$W = Fd.$$

Therefore,

$$W = (15)(8) = 120.$$

Hence,

$$\boxed{W = 120 \text{ J}}.$$

27.1.4 Example: Variable Force

Example 27.2.

A force

$$F(x) = 5x^{-2}$$

pounds acts on an object moving along a straight line. Find the work done in moving the object from

$$x = 1 \text{ ft} \quad \text{to} \quad x = 10 \text{ ft}.$$

Solution.

Using the work formula,

$$W = \int_1^{10} 5x^{-2}\,dx.$$

By integrating we get,

$$W = 5\left[-x^{-1}\right]_1^{10} = 5\left(-\frac{1}{10} + 1\right).$$

Thus,

$$W = 5\left(\frac{9}{10}\right) = \frac{9}{2}.$$

Therefore,

$$\boxed{W = \frac{9}{2} \text{ ft-lb}}.$$

27.1.5 Example: Work from a Force Graph

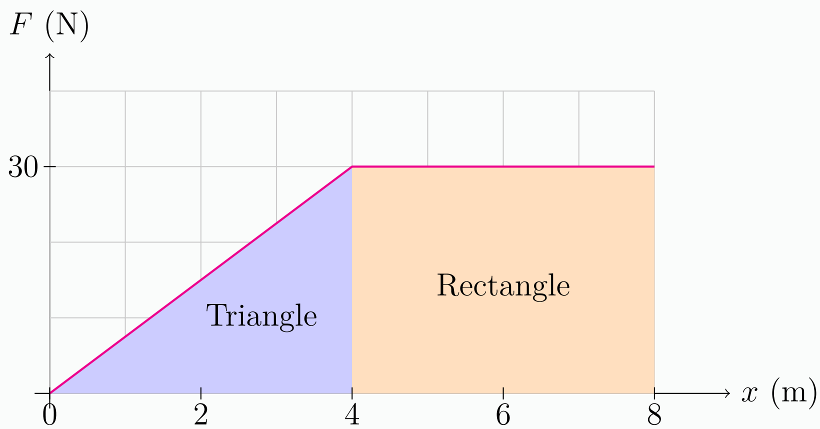
Example 27.3.

A force increases linearly from 0 N to 30 N over the first 4 m of motion and then remains constant at 30 N until the object has moved 8 m.

Find the work done.

Solution.

The work equals the area under the force graph.



The shaded area consists of:

$$\text{Triangle} + \text{Rectangle}.$$

Triangle:

$$A_1 = \frac{1}{2}(4)(30) = 60.$$

Rectangle:

$$A_2 = (4)(30) = 120.$$

Hence

$$W = A_1 + A_2 = 60 + 120 = 180.$$

Therefore,

$$\boxed{W = 180 \text{ J}}.$$

Important

Strategy for Work Problems

1. Identify the force function $F(x)$.
2. Determine the interval of motion $[a, b]$.
3. Set up
$$W = \int_a^b F(x)\,dx.$$
4. Evaluate the integral.
5. Include units.

27.2. Hooke's Law and Springs

One common application of work involves stretching or compressing a spring.

The farther a spring is stretched from its natural length, the more force is required to stretch it further.

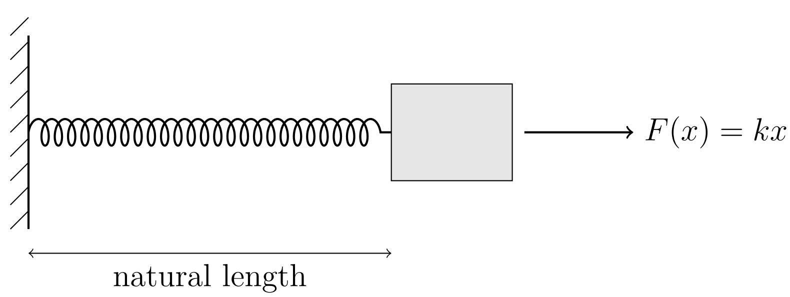
This relationship is described by **Hooke's Law**.

Important

Hooke's Law
If a spring is stretched x units beyond its natural length, then the force required to hold the spring stretched x units is
$$F(x) = kx.$$

Here:

$$\begin{aligned} F(x) &= \text{force} \\ x &= \text{distance stretched from natural length} \\ k &= \text{spring constant.} \end{aligned}$$



27.2.1 Work Done in Stretching a Spring

Since the force needed to stretch a spring changes with x, we use an integral.

If a spring is stretched from $x = a$ to $x = b$, then

$$W = \int_a^b F(x)\,dx.$$

Using Hooke's Law,

$$F(x) = kx.$$

Therefore,

$$W = \int_a^b kx\,dx.$$

Important

Work Done Stretching a Spring

If a spring has spring constant k, then the work required to stretch the spring from $x = a$ to $x = b$ is

$$\boxed{W = \int_a^b kx\,dx}$$

where x is measured from the natural length of the spring.

Warning

For spring problems, x measures the amount stretched beyond the natural length, not the total length of the spring.

Example 27.4.

A force of 40 N is required to hold a spring stretched 0.2 m beyond its natural length. Find the spring constant k.

Solution.

By Hooke's Law,

$$F(x) = kx.$$

We are given

$$F = 40 \qquad \text{when} \qquad x = 0.2.$$

Thus,

$$40 = k(0.2).$$

Solving for k,

$$k = \frac{40}{0.2} = 200.$$

Therefore,

$$\boxed{k = 200 \text{ N/m}}.$$

Example 27.5.

A spring has spring constant

$$k = 200 \text{ N/m}.$$

Find the work required to stretch the spring from its natural length to 0.3 m beyond its natural length.

Solution.

Using

$$W = \int_a^b kx\,dx,$$

we have

$$W = \int_0^{0.3} 200x\,dx.$$

$$W = 200\left[\frac{x^2}{2}\right]_0^{0.3}.$$

$$W = 100(0.3)^2.$$

$$W = 100(0.09) = 9.$$

Therefore,

$$\boxed{W = 9 \text{ J}}.$$

Example 27.6.

A spring has natural length 20 cm. A force of 12 N is needed to stretch it to a length of 26 cm. Find the work required to stretch the spring from 22 cm to 30 cm.

Solution.

First find the spring constant.
The spring is stretched from 20 cm to 26 cm, so the stretch is

$$26 - 20 = 6 \text{ cm} = 0.06 \text{ m}.$$

Using Hooke's Law,

$$F = kx.$$

Thus,

$$12 = k(0.06).$$

So

$$k = \frac{12}{0.06} = 200 \text{ N/m}.$$

Now convert the desired stretch distances.
Stretching from 22 cm means

$$22 - 20 = 2 \text{ cm} = 0.02 \text{ m}.$$

Stretching to 30 cm means

$$30 - 20 = 10 \text{ cm} = 0.10 \text{ m}.$$

Therefore,

$$W = \int_{0.02}^{0.10} 200x\,dx.$$

$$W = 200\left[\frac{x^2}{2}\right]_{0.02}^{0.10}.$$

$$W = 100\left[(0.10)^2 - (0.02)^2\right].$$

$$W = 100(0.01 - 0.0004).$$

$$W = 100(0.0096) = 0.96.$$

Thus,

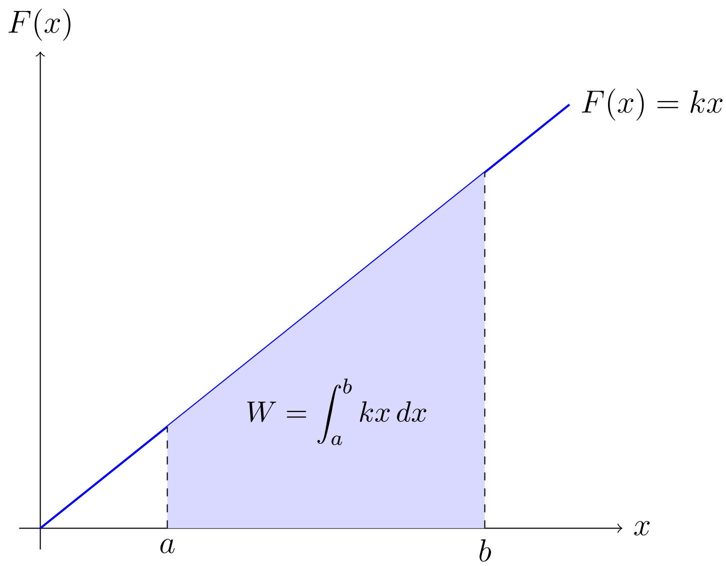
$$\boxed{W = 0.96 \text{ J}}.$$

27.2.2 Geometric Meaning for Springs

Since

$$F(x) = kx$$

is a line through the origin, the work done in stretching a spring is the area under this line.



27.3. Lifting Problems

Another important application of work involves lifting objects against gravity.

Recall that

$$\text{Work} = \text{Force} \times \text{Distance}.$$

When lifting an object vertically, the force required is equal to its weight.

Thus,

$$W = \text{Weight} \times \text{Distance}.$$

If every part of the object moves the same distance, then the work is easy to compute.

However, many objects (such as ropes, chains, and buckets) do not move as a single unit. Different parts move different distances, so integration is required.

Important

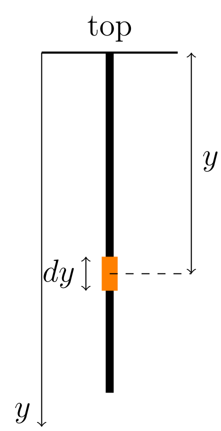
For lifting problems:

1. Draw a thin slice.
2. Determine the weight of the slice.
3. Determine the distance moved by the slice.
4. Compute
$$dW = (\text{weight})(\text{distance}).$$
5. Integrate over the entire object.

27.3.1 Lifting a Rope

Suppose a rope hangs vertically from a ceiling.

The top portions move only a short distance, while the lower portions must travel much farther.



Consider a thin slice of thickness dy located y units below the top.

If $w(y)$ denotes the weight density of the rope (weight per unit length), then the slice has weight
$$w(y)\,dy.$$

Since a small piece of rope located y units below the top must move a distance y, the slice must be lifted a distance y. Thus, the work done on the slice is

$$dW = w(y)\,y\,dy.$$

Example 27.7.

A rope is 20 ft long and weighs 3 lb/ft.
How much work is required to pull the entire rope to the top?

Solution.

Let y denote the distance below the top of the rope.
A small piece of rope of thickness dy has weight

$$dW_{\text{weight}} = 3\,dy.$$

Since the piece is y feet below the top, it must be lifted y feet.
Thus,

$$dW = (3\,dy)(y) = 3y\,dy.$$

Integrating over the entire rope,

$$W = \int_0^{20} 3y\,dy.$$

$$= 3\left[\frac{y^2}{2}\right]_0^{20}.$$

$$= \frac{3}{2}(400).$$

$$= 600.$$

Therefore,

$$\boxed{W = 600 \text{ ft-lb}}.$$

27.3.2 Lifting a Chain

Chains are treated exactly like ropes.

The only difference is that the weight density may be different.

Example 27.8.

A chain is 15 m long and weighs 4 N/m.
How much work is required to lift the entire chain to the top?

Solution.

Let y be the distance below the top.
A slice of thickness dy has weight

$$4\,dy.$$

The slice must be lifted y meters.
Therefore,

$$dW = (4\,dy)(y).$$

Thus,

$$W = \int_0^{15} 4y\,dy.$$

$$= 4\left[\frac{y^2}{2}\right]_0^{15}.$$

$$= 2(225).$$

$$= 450.$$

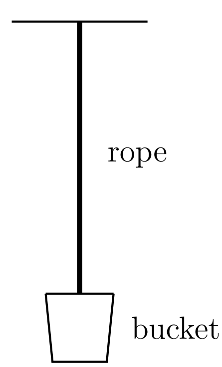
Hence,

$$\boxed{W = 450 \text{ J}}.$$

27.3.3 A Bucket and Rope

Now suppose a bucket is attached to a rope.

Both the bucket and the rope must be lifted.



Example 27.9.

A bucket weighing 40 lb is attached to a rope that weighs 2 lb/ft.
The rope is 30 ft long.
Find the work required to pull the bucket and rope to the top.

Solution.

Work done lifting the bucket

The bucket weighs 40 lb and is lifted 30 ft.

$$W_{\text{bucket}} = (40)(30) = 1200.$$

Work done lifting the rope

Let y denote the distance below the top.
A small piece of rope has weight

$$2\,dy.$$

It must be lifted y feet.
Thus,

$$dW = 2y\,dy.$$

Therefore,

$$W_{\text{rope}} = \int_0^{30} 2y\,dy.$$

$$= 2\left[\frac{y^2}{2}\right]_0^{30}.$$

$$= 900.$$

Hence,

$$W = W_{\text{bucket}} + W_{\text{rope}} = 1200 + 900.$$

$$\boxed{W = 2100 \text{ ft-lb}}$$

27.3.4 A Leaking Bucket

Sometimes a bucket is filled with liquid and lifted while the liquid leaks out.

In this case the weight changes continuously.

Example 27.10.

A bucket initially contains 24 lb of water.
It is lifted 30 ft to the top of a well.
Suppose water leaks out at a constant rate so that the bucket is empty when it reaches the top.
How much work is done lifting the water?

Solution.

Let x denote the distance lifted.
When $x = 0$,

$$\text{weight} = 24.$$

When $x = 30$,

$$\text{weight} = 0.$$

The weight decreases linearly.
Therefore,

$$w(x) = 24 - \frac{24}{30}x = 24 - \frac{4}{5}x.$$

A small displacement dx requires work

$$dW = w(x)\,dx.$$

Thus,

$$W = \int_0^{30} \left(24 - \frac{4}{5}x\right)\,dx.$$

$$= \left[24x - \frac{2}{5}x^2\right]_0^{30}.$$

$$= 720 - 360.$$

$$= 360.$$

Therefore,

$$\boxed{W = 360 \text{ ft-lb}}$$

Warning

In lifting problems, students often forget that different pieces move different distances. Always determine:

$$dW = (\text{weight of slice})(\text{distance moved}).$$

before integrating.

27.4. Pumping Liquids

One of the most important applications of work is pumping liquids from tanks.

The idea is the same as in lifting problems:

$$dW = \text{Weight} \times \text{Distance}.$$

The difference is that we divide the liquid into thin, horizontal slices.

Each slice:

- has a weight,
- is lifted a certain distance,
- contributes a small amount of work dW.

Adding all contributions gives the total work.

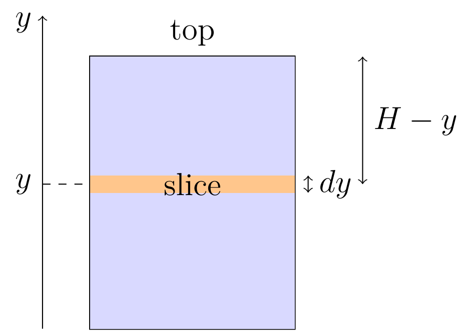
Important

Strategy for Pumping Problems

1. Draw a horizontal slice.
2. Find the volume of the slice.
3. Find the weight of the slice.
4. Determine how far the slice must be lifted.
5. Compute
$$dW = (\text{weight})(\text{distance}).$$
6. Integrate over the entire tank.

27.4.1 Rectangular Tank

Consider a rectangular tank filled with water.



A horizontal slice has thickness

$$dy.$$

Its volume is

$$dV = A(y)\,dy,$$

where $A(y)$ is the cross-sectional area.

Since water weighs approximately

$$62.4 \text{ lb/ft}^3,$$

the weight of the slice is

$$62.4\,A(y)\,dy.$$

Example 27.11.

A tank is 10 ft long, 4 ft wide, and 6 ft deep.
It is completely filled with water.
Find the work required to pump all the water to the top of the tank.

Solution.

Let y denote the height above the bottom.
A horizontal slice has volume

$$dV = (10)(4)\,dy = 40\,dy.$$

Its weight is

$$62.4(40)\,dy.$$

$$= 2496\,dy.$$

A slice at height y must be lifted $6 - y$ feet. Therefore,

$$dW = 2496(6 - y)\,dy.$$

By integrating from the bottom to the top we get,

$$W = \int_0^6 2496(6 - y)\,dy.$$

$$= 2496\left[6y - \frac{y^2}{2}\right]_0^6.$$

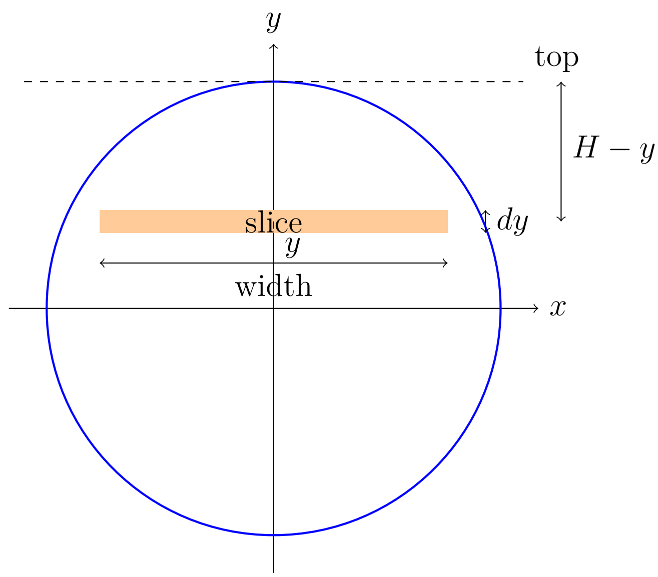
$$= 2496(36 - 18).$$

$$= 2496(18).$$

$$= \boxed{44{,}928 \text{ ft-lb}}.$$

27.4.2 Cylindrical Tank

For cylindrical tanks, the cross-sectional area changes with y. The geometry becomes more important.



A horizontal slice of liquid has thickness dy.

To compute the work required to pump the liquid from the tank, we focus on one slice at a time.

For each slice, we determine:

1. its volume,
2. its weight,
3. the distance it must be lifted.

The work required to move the slice is

$$dW = (\text{Weight of Slice}) \times (\text{Distance Moved}).$$

Adding the work done on all slices gives the total work:

$$W = \int dW.$$

The exact expression for dW depends on the shape of the tank and the fluid being pumped.

Example 27.12.

A cylindrical tank of radius 3 ft and length 10 ft is completely filled with water.
Find the work required to pump the water over the top edge of the tank.

Solution.

Let the origin be at the center of the circular end.
The circle is

$$x^2 + y^2 = 9.$$

At height y,

$$x = \sqrt{9 - y^2}.$$

The width of the slice is

$$2\sqrt{9 - y^2}.$$

Therefore,

$$A(y) = 10\left(2\sqrt{9 - y^2}\right).$$

$$= 20\sqrt{9 - y^2}.$$

The weight of the slice is

$$62.4\Big(20\sqrt{9 - y^2}\Big)\,dy.$$

A slice at height y must be lifted

$$3 - y$$

feet.
Thus,

$$W = \int_{-3}^{3} 62.4\Big(20\sqrt{9 - y^2}\Big)(3 - y)\,dy.$$

We leave the evaluation of this integral to technology.

Warning

The most common mistake in pumping problems is using the wrong distance.
Always ask:

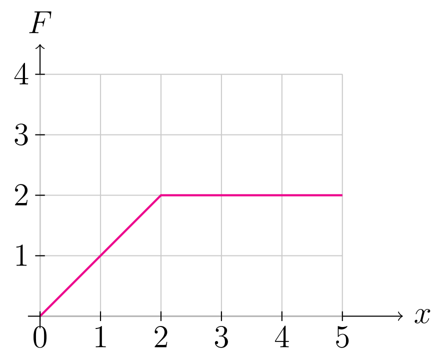
How far must this particular slice move?

Different slices usually travel different distances.

Exercises

Work

1. A constant force of 18 N moves an object 12 m. Find the work done.
2. A force
$$F(x) = 4x + 1$$
acts on an object moving from $x = 0$ to $x = 5$. Find the work done.
3. A force
$$F(x) = \cos\left(\frac{\pi x}{3}\right)$$
Newtons move an object from $x = 1$ m to $x = 2$ m. Compute the work done.
4. Use the graph below to estimate the work done by the force.



5. Explain why
$$W = \int_a^b F(x)\,dx$$
can be viewed as a limit of Riemann sums.

Hook'e Law

1. A spring has spring constant $k = 50$ N/m. Find the work required to stretch it from its natural length to 0.4 m beyond its natural length.
2. A spring has spring constant $k = 120$ N/m. Find the work required to stretch it from 0.1 m to 0.5 m beyond its natural length.
3. A force of 18 N is required to stretch a spring 0.06 m beyond its natural length. Find the spring constant.

4. A spring has natural length 15 cm. A force of 20 N stretches it to 19 cm. Find the work required to stretch it from 16 cm to 22 cm.

5. A spring has natural length 10 in. A force of 5 lb stretches it to 12 in. Find the work required to stretch it from 10 in to 15 in.

6. A force of 30 N stretches a spring 0.15 m. How much work is required to stretch it from 0.15 m to 0.25 m beyond its natural length?

Lifting Problems

1. A rope is 25 ft long and weighs 4 lb/ft. Find the work required to pull the entire rope to the top.

2. A chain is 12 m long and weighs 6 N/m. Find the work required to lift the chain completely.

3. A rope is 40 ft long and weighs 1.5 lb/ft. Find the work required to pull the rope onto a platform.

4. A bucket weighing 50 lb hangs from a 20-ft rope weighing 3 lb/ft. Find the work required to pull the bucket and rope to the top.

5. A bucket containing 30 lb of water is lifted 25 ft. The water leaks out at a constant rate and the bucket is empty at the top. Find the work done lifting the water.

6. A chain 10 m long weighs 5 N/m. How much work is required to lift only the lower half of the chain to the top?

7. A rope 50 ft long weighs 2 lb/ft. Find the work required to pull only the lower 20 ft of the rope to the top.

8. A bucket weighing 20 lb hangs from a 40-ft rope weighing 1 lb/ft. Find the work required to lift the bucket and rope.

Pumping Liquids

1. A rectangular tank is 8 ft long, 5 ft wide, and 4 ft deep. It is completely filled with water. Find the work required to pump all the water to the top.

2. A rectangular tank is 12 ft long, 6 ft wide, and 5 ft deep. Find the work required to pump the water over the top edge.

3. A cylindrical tank has radius 2 ft and length 8 ft. Find an integral representing the work required to pump all water over the top.

4. A cylindrical tank has radius 4 ft and length 15 ft. Set up (but do not evaluate) an integral for the work required to pump all water over the top.

5. A rectangular swimming pool is 30 ft long, 12 ft wide, and 6 ft deep. Find the work required to pump all water to ground level.

6. A tank contains oil weighing 55 lb/ft^3. The tank is 10 ft long, 4 ft wide, and 5 ft deep. Find the work required to pump all oil over the top.

7. Explain why

$$dW = (\text{weight})(\text{distance})$$

is the key idea behind all pumping problems.

Answers and Hints

Chapter 1: Algebra and Trigonometry Review

1. (a) Yes, the relation is a function.

   (b) No, the relation is not a function because the input -1 corresponds to two different outputs.

2.
$$\boxed{\text{Dom}(f) = \{-2, -1, 0, 2, 4\}}$$

$$\boxed{\text{Range}(f) = \{0, 1, 3, 5\}}$$

3. (a)
$$\boxed{f(2) = 9}$$

   (b)
$$\boxed{f(-1) = 6}$$

   (c)
$$\boxed{f(0) = 1}$$

4. (a)
$$\boxed{\{x \in \mathbb{R} : x \neq 3\}}$$

   or

$$\boxed{(-\infty, 3) \cup (3, \infty)}$$

   (b)
$$\boxed{\{x \in \mathbb{R} : x \geq 7\}}$$

   or

$$\boxed{[7, \infty)}$$

(c)

$$\boxed{\{x \in \mathbb{R} : x \geq -1,\ x \neq 2\}}$$

or

$$\boxed{[-1, 2) \cup (2, \infty)}$$

Hint. For an even root, the expression under the radical must be nonnegative. Also, the denominator cannot equal zero.

5. The domain is

$$\boxed{\{x \in \mathbb{R} : x \neq -3\}}$$

or

$$\boxed{(-\infty, -3) \cup (-3, \infty)}.$$

The range is

$$\boxed{\{y \in \mathbb{R} : y \neq 2\}}$$

or

$$\boxed{(-\infty, 2) \cup (2, \infty)}.$$

Hint. To find the range, let

$$y = \frac{2x - 1}{x + 3}$$

and solve for x in terms of y. Determine which value of y makes the resulting denominator zero.

6. (a)

$$\boxed{x^2 - 49 = (x - 7)(x + 7)}$$

(b)

$$\boxed{x^2 + 9x + 20 = (x + 4)(x + 5)}$$

(c)

$$\boxed{2x^2 + 7x + 3 = (2x + 1)(x + 3)}$$

(d)

$$\boxed{x^3 - 4x^2 + 5x - 20 = (x - 4)(x^2 + 5)}$$

Hint. For part (d), group the first two terms and the last two terms.

7. (a)

$$\boxed{x = 3,\ 4}$$

(b)

$$\boxed{x = -3,\ \frac{1}{2}}$$

(c) Over the complex numbers,

$$\boxed{x = -2 \pm i\sqrt{3}}.$$

Therefore, the equation has no real solutions.

8. (a)

$$\boxed{(-1, 6)}$$

(b)

$$\boxed{\left(-\infty, -\frac{5}{3}\right] \cup [1, \infty)}$$

Hint. Use

$$|A| < b \iff -b < A < b$$

and

$$|A| \geq b \iff A \leq -b \quad \text{or} \quad A \geq b.$$

9. (a)

$$\boxed{\frac{x^2 - 16}{x^2 + x - 20} = \frac{x+4}{x+5}}, \qquad x \neq 4, -5.$$

(b)

$$\boxed{\frac{x^2 + 7x + 12}{x^2 - 16} = \frac{x+3}{x-4}}, \qquad x \neq -4, 4.$$

Hint. Factor both the numerator and denominator completely before canceling common factors. Retain the restrictions from the original denominator.

10. (a)

$$\boxed{\frac{x+3}{3}}, \qquad x \neq 0.$$

(b)

$$\boxed{\frac{10 - x}{5}}, \qquad x \neq 0.$$

Hint. First combine the terms in the numerator into one fraction, and then multiply by the reciprocal of the denominator.

11. (a)

$$\boxed{\sqrt{72} = 6\sqrt{2}}$$

(b)

$$\boxed{\sqrt{147} = 7\sqrt{3}}$$

(c)

$$\boxed{\sqrt{48x^2} = 4|x|\sqrt{3}}$$

12. (a)

$$\boxed{\frac{5}{\sqrt{3}} = \frac{5\sqrt{3}}{3}}$$

(b)

$$\boxed{\frac{4}{2+\sqrt{5}} = 4\sqrt{5} - 8}$$

(c)

$$\boxed{\frac{\sqrt{7}}{\sqrt{2}} = \frac{\sqrt{14}}{2}}$$

Hint. In part (b), multiply the numerator and denominator by the conjugate $2 - \sqrt{5}$.

13. (a)

$$\boxed{a^8}$$

(b)

$$\boxed{x^5}, \qquad x \neq 0.$$

(c)

$$\boxed{8y^{12}}$$

(d)

$$\boxed{x^2}$$

14. (a)

$$\boxed{\frac{1}{2}}$$

(b)

$$\boxed{\frac{1}{2}}$$

(c)

$$\boxed{1}$$

(d)

$$\boxed{1}$$

15. (a)

$$\boxed{45^\circ = \frac{\pi}{4}}$$

(b)

$$\boxed{225^\circ = \frac{5\pi}{4}}$$

(c)

$$\boxed{\frac{7\pi}{6} = 210^\circ}$$

(d)

$$\boxed{\frac{5\pi}{3} = 300^\circ}$$

16. (a) The identity is verified:

$$\boxed{\frac{\sin x}{\cos x} = \tan x}.$$

(b) The identity is verified:

$$\boxed{1 + \tan^2 x = \sec^2 x}.$$

Hint. For part (a), use the quotient definition of tangent. For part (b), divide

$$\sin^2 x + \cos^2 x = 1$$

by $\cos^2 x$.

17. (a)

$$\boxed{x = \frac{\pi}{3} + 2n\pi \quad \text{or} \quad x = \frac{2\pi}{3} + 2n\pi, \qquad n \in \mathbb{Z}}$$

(b)

$$\boxed{x = 2n\pi \pm \frac{2\pi}{3}, \qquad n \in \mathbb{Z}}$$

(c)

$$\boxed{x = \frac{\pi}{4} + n\pi, \qquad n \in \mathbb{Z}}$$

Hint. First identify the reference angle and the quadrants in which the trigonometric function has the required sign. Then use the period of the function.

18. (a)

$$\boxed{x = \frac{\pi}{3}, \frac{2\pi}{3}}$$

(b)

$$\boxed{x = \frac{2\pi}{3}, \frac{4\pi}{3}}$$

(c)

$$\boxed{x = \frac{3\pi}{4}, \frac{7\pi}{4}}$$

Hint. Use the unit circle and include only the angles in the interval $0 \le x < 2\pi$.

19. (a) Yes, the graph of

$$y = |x|$$

represents a function.

(b) No, the graph of

$$x = y^2$$

does not represent y as a function of x. A vertical line with $x > 0$ intersects the graph at two points.

(c) Yes, the graph of

$$y = \sqrt{x}$$

represents a function.

Hint. Apply the Vertical Line Test: a graph represents a function if every vertical line intersects it at most once.

20. The square root requires $x - 1 \ge 0$. Thus, $x \ge 1$. The denominator also requires

$$x^2 - 9 \neq 0.$$

Hence, $x \neq -3$ and $x \neq 3$. Since $x \ge 1$, only $x = 3$ must be excluded. Therefore, in set-builder notation,

$$\boxed{\{x \in \mathbb{R} : x \ge 1,\ x \neq 3\}}$$

and in interval notation,

$$\boxed{[1, 3) \cup (3, \infty)}.$$

Hint. Combine the restriction from the square root with the restriction from the denominator.

Chapter 2: The Limit of a Function

1. 7.

   Hint: Direct substitution.

2. -27.

   Hint: Product Law.

3. $\frac{9}{5}$.

   Hint: Quotient Law.

4. 5.

   Hint: Root Law.

5. 10.

   Hint: Factor the numerator first.

6. $\frac{1}{6}$.

   Hint: Rationalize the numerator.

7. $$\lim_{x\to 2^-} f(x) = 3, \qquad \lim_{x\to 2^+} f(x) = 3, \qquad \lim_{x\to 2} f(x) = 3.$$

   Hint: Compute the one-sided limits separately.

8. $$\lim_{x\to 0^-} \frac{1}{x} = -\infty, \qquad \lim_{x\to 0^+} \frac{1}{x} = \infty, \qquad \lim_{x\to 0} \frac{1}{x} \text{ does not exist.}$$

9. 5.

   Hint: Use the Squeeze Theorem.

10. 4.

    Hint:
    $$\frac{\sin(4x)}{x} = 4 \cdot \frac{\sin(4x)}{4x}.$$

11. 0.

    Hint: Use
    $$1 - \cos x \approx \frac{x^2}{2} \quad \text{near} \quad 0, \quad \text{or rationalize.}$$

12. $\frac{1}{4}$.

    Hint: Rationalize the numerator.

Chapter 3: Continuity

1. Not continuous at $x = 2$.

   Hint: The function is not defined at $x = 2$, since the denominator is zero. However,

   $$\lim_{x\to 2} \frac{x^2 - 4}{x - 2} = 4.$$

2. Continuous at $x = 0$.

   Hint:

   $$\lim_{x\to 0} |x| = 0 = f(0).$$

3. Continuous on

   $$[-3, 3].$$

   Hint: We need

   $$9 - x^2 \geq 0.$$

4. $$\lim_{x\to 0} \cos(x^2) = 1.$$

   Hint: Use direct substitution.

5. Let

   $$f(x) = x^3 - 2x - 5.$$

   Then

   $$f(2) = 8 - 4 - 5 = -1$$

   and

   $$f(3) = 27 - 6 - 5 = 16.$$

   Since f is continuous and changes sign on $[2, 3]$, the Intermediate Value Theorem guarantees a solution between 2 and 3.

6. Continuous nowhere.

   Hint: Near every real number, there are both rational and irrational numbers, so the function jumps between 0 and 1.

7. Continuous only at

   $$x = 0.$$

Hint: At $x = 0$, both rational and irrational approaches give values approaching 0. At any $x \neq 0$, rational and irrational values approach different limits.

8. Yes.

   Hint: We want
$$x = x^3 + 1.$$
   Equivalently,
$$x^3 - x + 1 = 0.$$
   Let
$$f(x) = x^3 - x + 1.$$
   Since
$$f(-2) = -8 + 2 + 1 = -5$$
   and
$$f(-1) = -1 + 1 + 1 = 1,$$
   there is a solution in $(-2, -1)$.

9. Continuous on
$$(-\infty, \infty).$$
   Hint: For $x \neq 0$, the function is continuous. At $x = 0$, use
$$-1 \leq \sin(1/x) \leq 1.$$
   Thus,
$$-x^4 \leq x^4 \sin(1/x) \leq x^4.$$
   Since
$$\lim_{x\to 0}(-x^4) = 0 \qquad \text{and} \qquad \lim_{x\to 0} x^4 = 0,$$
   the Squeeze Theorem gives
$$\lim_{x\to 0} x^4 \sin(1/x) = 0 = f(0).$$

10. Hint:

    Let $u(t)$ be the monk's distance from the monastery on the first day at time t, and let $v(t)$ be the monk's distance from the monastery on the second day at time t.

    Define
$$h(t) = u(t) - v(t).$$

At 7:00 AM,

$$h(t) < 0,$$

because the first monk's position is at the monastery and the second-day position is at the top.

At 7:00 PM,

$$h(t) > 0.$$

Since h is continuous, the Intermediate Value Theorem implies that for some time t,

$$h(t) = 0.$$

So the monk is at the same point on the path at the same time of day on both days.

Chapter 4: Derivatives and Rates of Change

1.

$$\frac{f(4) - f(1)}{4 - 1} = \frac{17 - 2}{3} = 5.$$

Hint: Average rate of change is

$$\frac{f(b) - f(a)}{b - a}.$$

2.

$$m = 2.$$

Hint: Use

$$m = \lim_{h \to 0} \frac{f(1+h) - f(1)}{h}.$$

So, the tangent line slope at $x = 1$ is 2.

3.

$$v(t) = s'(t) = 2t - 4.$$

Thus,

$$v(3) = 2.$$

4.

$$y' = 2x - 3.$$

At $x = 1$,

$$m = -1.$$

Using the point $(1, 0)$,

$$y = -x + 1.$$

5.

$$f'(4) = \frac{1}{4}.$$

Hint:

$$f'(4) = \lim_{h \to 0} \frac{\sqrt{4+h} - 2}{h}.$$

Rationalize the numerator.

Chapter 5: The Derivative as a Function

1.

$$f'(x) = 6 - 8x.$$

$$\mathrm{Dom}(f) = (-\infty, \infty), \qquad \mathrm{Dom}(f') = (-\infty, \infty).$$

Hint: Use

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$

2.

$$f'(x) = 2x - 9x^2.$$

$$\mathrm{Dom}(f) = (-\infty, \infty), \qquad \mathrm{Dom}(f') = (-\infty, \infty).$$

3.

$$H'(t) = \frac{-6}{(4+t)^2}.$$

$$\mathrm{Dom}(H) = (-\infty, -4) \cup (-4, \infty),$$

$$\mathrm{Dom}(H') = (-\infty, -4) \cup (-4, \infty).$$

4.

$$f'(x) = \frac{-1}{(x+2)^2}.$$

$$\mathrm{Dom}(f) = (-\infty, -2) \cup (-2, \infty),$$

$$\mathrm{Dom}(f') = (-\infty, -2) \cup (-2, \infty).$$

5.

$$f'(x) = \frac{1}{2\sqrt{x+1}}.$$

$$\mathrm{Dom}(f) = [-1, \infty), \qquad \mathrm{Dom}(f') = (-1, \infty).$$

Hint: Rationalize the numerator when using the definition.

6.
$$f(x) = |x - 3|$$
is differentiable for
$$x \neq 3.$$
It is not differentiable at $x = 3$ because there is a corner.

7.
$$f(x) = x^{2/3}$$
is differentiable for
$$x \neq 0.$$
It is not differentiable at $x = 0$ because the tangent line is vertical.

8.
$$g(x) = x + |x| = \begin{cases} 0, & x < 0, \\ 2x, & x \geq 0. \end{cases}$$
So g is differentiable for
$$x \neq 0.$$

$$g'(x) = \begin{cases} 0, & x < 0, \\ 2, & x > 0. \end{cases}$$
It is not differentiable at $x = 0$.

9.
$$f'(x) = 4x^3 - 10x + 2.$$

$$f''(x) = 12x^2 - 10.$$

10.
$$f'(x) = 5x^4 - 12x^2 + 1; \qquad f''(x) = 20x^3 - 24x; \qquad f'''(x) = 60x^2 - 24.$$

11. The graph is not differentiable at approximately
$$x = -1 \qquad \text{and} \qquad x = 1.5.$$

Hint: At these points, the pieces of the graph do not connect smoothly. There are jumps or sharp changes in behavior.

12.

$$f''(1) \text{ is bigger.}$$

Hint: $f'(-1)$ is the value of the derivative graph at $x = -1$. $f''(1)$ is the slope of the derivative graph at $x = 1$.

13. The graph is not differentiable at

$$x = -1 \qquad \text{and} \qquad x = 1.$$

Hint: The function is not continuous at these points, so it cannot be differentiable there.

14. The graph is not differentiable at

$$x = -1, \qquad x = 0, \qquad x = 2.$$

Hint:

- At $x = -1$, there is a corner.
- At $x = 0$, there is a jump.
- At $x = 2$, the pieces do not connect smoothly.

15.

$$f'(-1) \text{ is bigger.}$$

Hint: From the formulas shown in the graph,

$$f'(-1) = 0 \qquad \text{while} \qquad f''(1) < 0.$$

16.

$$f''(0) \text{ is bigger.}$$

Hint:

$$f'(-1) < 0, \qquad \text{but} \qquad f''(0) = 0.$$

Chapter 6: Differentiation Formulas

1. (i).

$$\boxed{f'(x) = 36x^3 - 10x}$$

(ii).

$$\boxed{g'(t) = -12t^{-4} + 3t^{-1/2}}$$

Equivalently,

$$\boxed{g'(t) = -\frac{12}{t^4} + \frac{3}{\sqrt{t}}}.$$

(iii).

$$\boxed{h'(x) = 15x^2 - 4x + 8}$$

(iv).

$$\boxed{f'(x) = 6x - 12x^2}$$

(v).

$$\boxed{g'(t) = 6t^2 + 2t - 10}$$

(vi).

$$\boxed{y' = -\frac{8x}{(x^2-1)^2}}$$

(vii). First simplify:

$$f(x) = \frac{x}{x + \frac{2}{x}} = \frac{x^2}{x^2+2}, \qquad x \neq 0.$$

Therefore,

$$\boxed{f'(x) = \frac{4x}{(x^2+2)^2}}, \qquad x \neq 0.$$

Hint. Multiply the numerator and denominator of the original expression by x before differentiating.

(viii). Rewrite the function using powers:

$$y = x^{3/2} + 2x^{1/2} + x^{-1/2}.$$

Thus,

$$\boxed{y' = \frac{3}{2}x^{1/2} + x^{-1/2} - \frac{1}{2}x^{-3/2}}$$

or

$$\boxed{y' = \frac{3}{2}\sqrt{x} + \frac{1}{\sqrt{x}} - \frac{1}{2x^{3/2}}}, \qquad x > 0.$$

Hint. Divide each term in the numerator by $\sqrt{x}$, and then apply the Power Rule.

(ix). Expanding first gives

$$u = t^{-2} + 2t^{-3/2} + t^{-1}.$$

Therefore,

$$\boxed{\frac{du}{dt} = -2t^{-3} - 3t^{-5/2} - t^{-2}}$$

or

$$\boxed{\frac{du}{dt} = -\frac{2}{t^3} - \frac{3}{t^{5/2}} - \frac{1}{t^2}}, \qquad t > 0.$$

Hint. First expand the square and rewrite all radicals using fractional exponents.

2. Using the Product Rule,

$$f'(x) = (2x)(x - 3x^2) + (1 + x^2)(1 - 6x).$$

After simplifying,

$$\boxed{f'(x) = 1 - 6x + 3x^2 - 12x^3}.$$

Alternatively, expanding first gives

$$f(x) = x - 3x^2 + x^3 - 3x^4,$$

and therefore,

$$\boxed{f'(x) = 1 - 6x + 3x^2 - 12x^3}.$$

3. The derivative is

$$y' = \frac{3(x+2) - 3x}{(x+2)^2} = \frac{6}{(x+2)^2}.$$

At $x = 2$,

$$y'(2) = \frac{6}{(2+2)^2} = \frac{3}{8}.$$

Thus, the tangent line has slope $\frac{3}{8}$ and passes through $\left(2, \frac{3}{2}\right)$. Hence,

$$\boxed{y - \frac{3}{2} = \frac{3}{8}(x - 2)}$$

or, equivalently,

$$\boxed{y = \frac{3}{8}x + \frac{3}{4}}.$$

4. The expression inside the absolute value is zero when

$$x^2 - 9 = 0,$$

so

$$x = -3 \qquad \text{or} \qquad x = 3.$$

The function has sharp corners at these two numbers. Therefore, g is differentiable for

$$\boxed{(-\infty,-3)\cup(-3,3)\cup(3,\infty)}.$$

Equivalently,

$$\boxed{g \text{ is differentiable for all } x\in\mathbb{R} \text{ except } x=-3 \text{ and } x=3.}$$

Hint. Write the absolute-value function in piecewise form and check the points where $x^2-9=0$.

5. The given line can be written as
$$y = x - 5,$$
so its slope is 1.

For the curve,
$$y = \frac{x+3}{x-2},$$
we have
$$y' = \frac{(x-2)-(x+3)}{(x-2)^2} = -\frac{5}{(x-2)^2}.$$
Since
$$-\frac{5}{(x-2)^2} < 0$$
for every $x \neq 2$, the derivative can never equal 1.

Therefore,
$$\boxed{\text{There are no tangent lines parallel to } x-y=5.}$$

Hint. Parallel lines must have equal slopes. Set the derivative of the curve equal to the slope of the given line.

6. The only possible points of nondifferentiability are the boundary points $x=-1$ and $x=3$.

At $x=-1$,
$$f(-1)=0,$$
but
$$\lim_{x\to -1^+} f(x) = 1.$$

Thus, f is discontinuous and therefore not differentiable at $x=-1$.

At $x = 3$,

$$\lim_{x \to 3^-} f(x) = 9,$$

but

$$f(3) = 2.$$

Thus, f is discontinuous and therefore not differentiable at $x = 3$.

Therefore, f is differentiable for

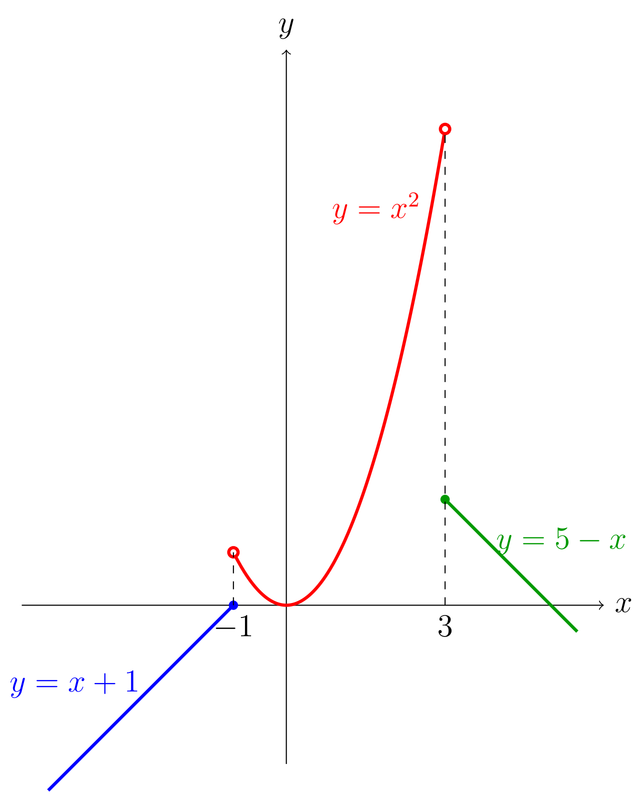
$$\boxed{(-\infty, -1) \cup (-1, 3) \cup (3, \infty)}.$$

(a)

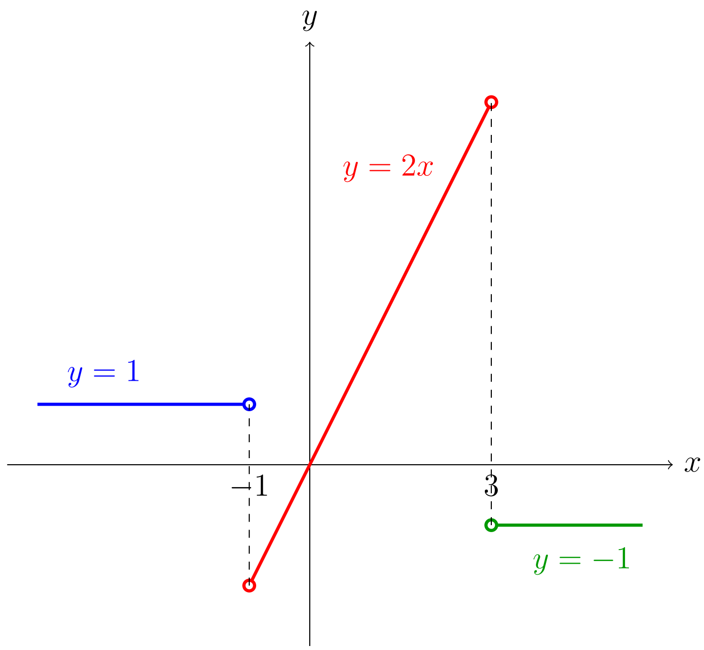
$$\boxed{f'(x) = \begin{cases} 1, & x < -1, \\ 2x, & -1 < x < 3, \\ -1, & x > 3. \end{cases}}$$

The derivative is undefined at $x = -1$ and $x = 3$.

(b) The graphs of f and f' are shown below.



Graph of f



Graph of f'

Hint. A function must be continuous at a point in order to be differentiable there. First check continuity at $x = -1$ and $x = 3$.

7. Since

$$R(p) = pf(p),$$

the Product Rule gives

$$R'(p) = f(p) + pf'(p).$$

(a)

$$\boxed{f(25) = 8000}$$

means that when the price is \$25 per item, the company sells 8000 items.

(b)

$$\boxed{f'(25) = -200}$$

means that when the price is near \$25, increasing the price by \$1 decreases the number of items sold by approximately 200.

(c)

$$\begin{aligned} R'(25) &= f(25) + 25f'(25) \\ &= 8000 + 25(-200) \\ &= 8000 - 5000 \\ &= 3000. \end{aligned}$$

Therefore,

$$\boxed{R'(25) = 3000}.$$

This means that when the price is near \$25, increasing the price by \$1 increases the total revenue by approximately \$3000.

Hint. Differentiate $R(p) = pf(p)$ using the Product Rule.

Chapter 7: Derivatives of Trigonometric Functions

1. (i).

$$\boxed{\lim_{x\to 0} \frac{\sin(7x)}{x} = 7}$$

Hint. Write

$$\frac{\sin(7x)}{x} = 7\frac{\sin(7x)}{7x}.$$

(ii).

$$\boxed{\lim_{x\to 0} \frac{\tan(4x)}{x} = 4}$$

Hint. Write

$$\frac{\tan(4x)}{x} = 4\frac{\tan(4x)}{4x}.$$

(iii).

$$\boxed{\lim_{x\to 0} \frac{1-\cos x}{x^2} = \frac{1}{2}}$$

Hint. Use

$$1 - \cos x = 2\sin^2\left(\frac{x}{2}\right).$$

(iv).

$$\boxed{\lim_{x\to 0} \frac{\sin(2x)}{\tan(5x)} = \frac{2}{5}}$$

Hint. Rewrite the expression as

$$\frac{\sin(2x)}{2x} \cdot \frac{5x}{\tan(5x)} \cdot \frac{2}{5}.$$

(v).

$$\boxed{\lim_{x\to 0} \frac{\sin(x^3)}{x} = 0}$$

Hint. Write

$$\frac{\sin(x^3)}{x} = \frac{\sin(x^3)}{x^3}\, x^2.$$

(vi).

$$\boxed{\lim_{x\to\pi/3} \frac{\sin x - \sqrt{3}/2}{x - \pi/3} = \frac{1}{2}}$$

Hint. Recognize the limit as the derivative of

$$f(x) = \sin x$$

at $x = \frac{\pi}{3}$.

2. (i).

$$\boxed{f'(x) = 3x^2 \cos x - x^3 \sin x}$$

(ii).

$$\boxed{y' = \tan x + x \sec^2 x - \sec x \tan x}$$

(iii).

$$\boxed{y' = \frac{1}{1 + \cos x}}$$

Hint. Apply the Quotient Rule and use

$$\sin^2 x + \cos^2 x = 1.$$

(iv).

$$\boxed{y' = \frac{\cos x - x(1+x)\sin x}{(1+x)^2}}$$

Hint. Apply the Product Rule to $x \cos x$, followed by the Quotient Rule.

3. For

$$y = \sin x - x \cos x,$$

we have

$$y' = \cos x - (\cos x - x \sin x) = x \sin x.$$

Thus,

$$y'(0) = 0.$$

Therefore, the tangent line at $(0,0)$ is

$$\boxed{y = 0}.$$

4. Given

$$G(t) = t \cos t,$$

the first derivative is

$$\boxed{G'(t) = \cos t - t \sin t}.$$

The second derivative is

$$\boxed{G''(t) = -2 \sin t - t \cos t}.$$

5. Since

$$g(x) = f(x) \cos x,$$

the Product Rule gives

$$g'(x) = f'(x) \cos x - f(x) \sin x.$$

Therefore,

$$\begin{aligned} g'\left(\frac{\pi}{6}\right) &= f'\left(\frac{\pi}{6}\right) \cos\left(\frac{\pi}{6}\right) - f\left(\frac{\pi}{6}\right) \sin\left(\frac{\pi}{6}\right) \\ &= 1 \cdot \frac{\sqrt{3}}{2} - 3 \cdot \frac{1}{2}. \end{aligned}$$

Hence,

$$\boxed{g'\left(\frac{\pi}{6}\right) = \frac{\sqrt{3} - 3}{2}}.$$

6. For

$$f(x) = 2x - \sin x,$$

we have

$$f'(x) = 2 - \cos x.$$

A horizontal tangent would require

$$2 - \cos x = 0,$$

or

$$\cos x = 2.$$

Since

$$-1 \le \cos x \le 1,$$

this equation has no real solution.

Therefore,

$$\boxed{\text{The graph has no horizontal tangent.}}$$

Hint. Set $f'(x) = 0$ and determine whether the resulting trigonometric equation has any real solutions.

Chapter 8: The Chain Rule

1.
$$y' = 30(5x - 1)^5.$$

Hint: Chain Rule.

2.
$$y' = \frac{1}{\sqrt{2x + 7}}.$$

Hint: Rewrite as $(2x + 7)^{1/2}$.

3.
$$y' = 4\cos(4x).$$

4.
$$y' = -3x^2 \sin(x^3).$$

5.
$$y' = \frac{\sec^2(\sqrt{x})}{2\sqrt{x}}.$$

6.
$$y' = -6x(x^2 + 1)^{-4}.$$

7.
$$y' = 7(3x^2 - 5x)^6(6x - 5).$$

8.
$$y' = 2x \sec(x^2) \tan(x^2).$$

9.
$$y' = \frac{\cos x}{2\sqrt{1 + \sin x}}.$$

10.
$$y' = -5 \sin x(1 + \cos x)^4.$$

11.
$$y' = 4\left(x + \frac{1}{x}\right)^3 \left(1 - \frac{1}{x^2}\right).$$

12.
$$y' = -\frac{1}{(x-1)^2\sqrt{\frac{x+1}{x-1}}}.$$

Hint: Let
$$u = \frac{x+1}{x-1}.$$

13.
$$y' = -\sin x \cos(\cos x).$$

14.
$$y' = -\csc^2(\sin x) \cos x.$$

15.
$$y' = \sec^2(\sec x) \sec x \tan x.$$

16. At $x = 0$,
$$y' = 6(2x + 1)^2, \qquad m = 6.$$

Tangent line:
$$y = 6x + 1.$$

17.
$$y' = \frac{1}{2\sqrt{x + 4}}.$$

At $x = 5$,

$$m = \frac{1}{6}.$$

Tangent line:

$$y - 3 = \frac{1}{6}(x - 5).$$

18.

$$y' = 2x\cos(x^2).$$

At $x = 0$,

$$m = 0.$$

Tangent line:

$$y = 0.$$

19.

$$F'(x) = f'(g(x))g'(x).$$

Thus,

$$F'(1) = f'(g(1))g'(1) = f'(2)(4) = 20.$$

20.

$$r'(x) = f'(g(h(x)))g'(h(x))h'(x).$$

Thus,

$$r'(1) = f'(g(h(1)))g'(h(1))h'(1).$$

Since

$$h(1) = 3, \qquad g(3) = 5,$$

we get

$$r'(1) = f'(5)g'(3)h'(1) = 6 \cdot 4 \cdot 2 = 48.$$

Chapter 9: Implicit Differentiation

1.
$$\frac{dy}{dx} = -\frac{2x + y}{x + 2y}.$$

Hint: Differentiate each term with respect to x, remembering that

$$\frac{d}{dx}(y^2) = 2y\frac{dy}{dx}.$$

2.
$$\frac{dy}{dx} = \frac{3x - y}{x - y}.$$

3.
$$\frac{dy}{dx} = -\frac{2x(2x^2 + y^2)}{y(2x^2 + 3y)}.$$

4.
$$\frac{dy}{dx} = \frac{x(x + 2y)}{x^2 + 2y(x + y)^2}.$$

Hint: Use the Quotient Rule on

$$\frac{x^2}{x + y}.$$

5.
$$\frac{dy}{dx} = \frac{\sqrt{x + y} - 8x^3\sqrt{x + y}}{8y^3\sqrt{x + y} - 1}.$$

Hint: Differentiate

$$\sqrt{x + y} = (x + y)^{1/2}$$

using the Chain Rule.

6.
$$\frac{dy}{dx} = \frac{\sin(y^2) - 2xy\cos(x^2)}{\sin(x^2) - 2xy\cos(y^2)}.$$

7.
$$\frac{dy}{dx} = \frac{y\sec^2(x/y) - y^2}{x\sec^2(x/y) + y^2}.$$

Hint:
$$\frac{d}{dx}\left(\frac{x}{y}\right) = \frac{y - xy'}{y^2}.$$

8.
$$\frac{dy}{dx} = -\frac{y\cos(xy) + \sin(x+y)}{x\cos(xy) + \sin(x+y)}.$$

9.
$$\frac{dy}{dx} = \frac{(1+x^2)^2\sec^2(x-y) + 2xy}{(1+x^2)^2\sec^2(x-y) + (1+x^2)}.$$

10.
$$\frac{dx}{dy} = -\frac{x(2x^3y - x^2 + 6y^2)}{y(4x^3y - 3x^2 + 2y^2)}.$$

11.
$$\frac{dx}{dy} = \frac{x\sec x - \sec y}{y\sec x\tan x - \tan y}.$$

12. The slope at
$$\left(\frac{\pi}{2}, \frac{\pi}{4}\right)$$
is
$$m = \frac{1}{2}.$$
Tangent line:
$$y - \frac{\pi}{4} = \frac{1}{2}\left(x - \frac{\pi}{2}\right).$$

13. The slope at (π, π) is
$$m = \frac{1}{3}.$$
Tangent line:
$$y - \pi = \frac{1}{3}(x - \pi).$$

14. The slope at $(2, 1)$ is
$$m = \frac{3}{4}.$$
Tangent line:
$$y - 1 = \frac{3}{4}(x - 2).$$

15. The slope at $(2, 1)$ is
$$m = -\frac{1}{2}.$$
Tangent line:
$$y - 1 = -\frac{1}{2}(x - 2).$$

16. The slope at

$$\left(0, \frac{1}{2}\right)$$

is

$$m = 1.$$

Tangent line:

$$y - \frac{1}{2} = x.$$

17.

$$y'' = -\frac{x^2 + 4y^2}{16y^3}.$$

Since

$$x^2 + 4y^2 = 4,$$

this can also be written as

$$y'' = -\frac{1}{4y^3}.$$

18.

$$y'' = -\frac{6(x^2 + xy + y^2)}{(x + 2y)^3}.$$

Since

$$x^2 + xy + y^2 = 3,$$

this can also be written as

$$y'' = -\frac{18}{(x + 2y)^3}.$$

19.

$$y'' = \frac{\sin^2 x \sin y + \cos x \cos^2 y}{\cos^3 y}.$$

20.

$$y'' = \frac{6(x^2 - xy + y^2)}{(x - 2y)^3}.$$

Since

$$x^2 - xy + y^2 = 7,$$

this can also be written as

$$y'' = \frac{42}{(x - 2y)^3}.$$

21.

$$f'(1) = -\frac{16}{13}.$$

Hint: Differentiate

$$f(x) + x^2[f(x)]^3 = 10$$

implicitly with respect to x, then use $f(1) = 2$.

22.

$$g'(0) = 0.$$

Hint: First note that

$$g(0) = 0.$$

23.

$$F'(5) = 24.$$

Hint:

$$F'(x) = f'(g(x))g'(x).$$

24.

$$r'(1) = 120.$$

Hint:

$$r'(x) = f'(g(h(x)))g'(h(x))h'(x).$$

Chapter 10: Rates of Change in the Natural and Social Sciences

1.

$$v(t) = 3t^2 - 12, \qquad a(t) = 6t.$$

2.

$$v(t) = 3t^2 - 18t + 24 = 3(t-2)(t-4).$$

At rest when

$$t = 2, \quad t = 4.$$

3. Moving in the positive direction when

$$v(t) > 0.$$

Thus,

$$(0, 2) \cup (4, \infty).$$

4.

$$a(t) = 6t - 18 = 6(t-3).$$

Speeding up on

$$(2,3) \cup (4,\infty).$$

Slowing down on

$$(0,2) \cup (3,4).$$

5.

$$v(t) = 6t^2 - 30t + 24 = 6(t-1)(t-4).$$

Check $t = 0, 1, 4, 5$:

$$s(0) = 0, \quad s(1) = 11, \quad s(4) = -16, \quad s(5) = -5.$$

Total distance:

$$11 + 27 + 11 = \boxed{49}.$$

6.

$$P'(t) = 60 + 4t.$$

$$P'(10) = 100.$$

Growth rate: $\boxed{100}$ fish per year.

7.

$$m'(t) = 0.5 + 0.06t.$$

$$m'(8) = 0.98.$$

8.

$$V'(t) = -100\left(1 - \frac{t}{40}\right).$$

$$V'(10) = -75.$$

Water drains at $\boxed{75}$ units per minute.

9.

$$v(t) = 96 - 32t.$$

Set $v(t) = 0$:

$$t = 3.$$

Maximum height:

$$s(3) = 144.$$

10. Solve

$$64t - 16t^2 = 48.$$

Then

$$t = 1, \quad t = 3.$$

Velocity:

$$v(t) = 64 - 32t.$$

So

$$v(1) = 32, \qquad v(3) = -32.$$

11.

$$C'(x) = 4 + 0.02x.$$

12.

$$C'(200) = 4 + 0.02(200) = 8.$$

13.

$$R'(x) = 50 - 0.1x.$$

14.

$$P'(x) = 500 - 0.4x.$$

15.

$$P'(t) = 120$$

means the population is increasing at a rate of 120 individuals per unit time at time t.

16. From the velocity graph:

Speeding up when $v(t)$ and $v'(t)$ have the same sign.

Approximately,

$$\text{speeding up on } (1,2) \cup (3,4).$$

$$\text{slowing down on } (0,1) \cup (2,3).$$

Hint: Since this is a velocity graph, $a(t) = v'(t)$ is the slope of the graph.

17. The velocity is positive on the whole interval shown.

So the particle speeds up when the velocity graph is increasing, and slows down when the velocity graph is decreasing.

Approximately,

$$\text{speeding up on } (0,1) \cup (3,4),$$

$$\text{slowing down on } (1,3).$$

Chapter 11: Related Rates

1. $\boxed{80 \text{ cm}^2/\text{s}}$

   Hint: Use $A = s^2$, so
$$\frac{dA}{dt} = 2s\frac{ds}{dt}.$$

2. $\boxed{30\pi \text{ cm}^2/\text{min}}$

   Hint: Use $A = \pi r^2$.

3. $\boxed{800\pi \text{ mm}^3/\text{s}}$

   Hint: Use
$$V = \frac{4}{3}\pi r^3.$$

4. $\boxed{\frac{5}{36\pi} \text{ m/min}}$

   Hint: Use
$$V = \pi r^2 h = 36\pi h.$$

5. The water is leaving at:
$$\boxed{150} \quad \text{when } t = 10,$$
$$\boxed{100} \quad \text{when } t = 20,$$
$$\boxed{50} \quad \text{when } t = 30.$$

   Hint:
$$V'(t) = -200\left(1 - \frac{t}{40}\right).$$

6.
$$\boxed{\frac{dy}{dt} = \frac{3}{2}.}$$

   Hint: Differentiate
$$y = \sqrt{3x+1}$$
   with respect to t.

7.
$$\boxed{\frac{dy}{dt} = -\frac{9}{4}.}$$

Hint: Differentiate
$$x^2 + y^2 = 100.$$

8.
$$\boxed{\frac{dx}{dt} = -\frac{3}{4}.}$$

Hint: Differentiate
$$4x^2 + 9y^2 = 36.$$

9. No such value exists with the given information.

Hint: Differentiating gives
$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} + 2z\frac{dz}{dt} = 0.$$

At $(3, 4, 0)$,
$$2z\frac{dz}{dt} = 0,$$
but
$$2(3)(2) + 2(4)(1) = 20 \neq 0.$$

So the given rates are inconsistent with the constraint at that point.

10.
$$\boxed{\frac{dy}{dt} = -\frac{5}{6}\ \text{ft/s.}}$$

The top of the ladder is moving downward at
$$\boxed{\frac{5}{6}\ \text{ft/s.}}$$

Hint: Use
$$x^2 + y^2 = 13^2.$$

11.
$$\boxed{\frac{48}{7}\ \text{ft/s.}}$$

Hint: Use similar triangles. If x is the distance from the pole to the person and y is the shadow length, then

$$\frac{12}{x+y} = \frac{5}{y}.$$

12.

$$\boxed{-\frac{25}{\sqrt{145}} \text{ km/h.}}$$

The negative sign means the distance is decreasing.

Hint: After 3 hours,

$$x = 40, \qquad y = 45.$$

Use

$$z^2 = x^2 + y^2.$$

13.

$$\boxed{\frac{18}{5} \text{ ft/s.}}$$

Hint: If x is the horizontal distance and s is the string length, then

$$s^2 = x^2 + 80^2.$$

When $s = 100$, we get $x = 60$.

14.

$$\boxed{-\frac{3}{125} \text{ rad/s.}}$$

Hint: Use

$$\tan\theta = \frac{120}{x}.$$

When the string length is 200, the horizontal distance is

$$x = 160.$$

15.

$$\boxed{60\sqrt{3} \text{ cm}^2\text{/min.}}$$

Hint: For an equilateral triangle,

$$A = \frac{\sqrt{3}}{4}s^2.$$

16.

$$\boxed{120\pi\ \text{cm}^2/\text{min.}}$$

Hint: Use

$$S = 4\pi r^2.$$

17.

$$\boxed{\frac{dy}{dt} = \pi\sqrt{3}.}$$

Hint: Differentiate

$$y = 2\sin\left(\frac{\pi x}{2}\right)$$

with respect to t.

18.

$$\boxed{35} \quad \text{after 5 hours,}$$

$$\boxed{20} \quad \text{after 20 hours.}$$

Hint:

$$P'(t) = 40 - t.$$

19.

$$\boxed{16\ \text{grams/day.}}$$

Hint:

$$M'(t) = 12 + 0.4t.$$

20.

$$\boxed{C'(x) = 4 + 0.02x.}$$

At $x = 200$,

$$\boxed{C'(200) = 8.}$$

Chapter 12: Linear Approximations and Differentials

1.

$$L(x) = 0 + (1)(x - 1) = x - 1.$$

Hint:

$$L(x) = f(a) + f'(a)(x - a).$$

2.

$$L(x) = 3 + \frac{1}{6}(x - 9).$$

Hint:

$$f'(x) = \frac{1}{2\sqrt{x}}.$$

3.

$$L(x) = \frac{1}{2} + \frac{\sqrt{3}}{2}\left(x - \frac{\pi}{6}\right).$$

Hint:

$$f'(x) = \cos x.$$

4.

$$\sqrt{15.8} \approx 3.975.$$

Hint: Use $f(x) = \sqrt{x}$ near $a = 16$.

5.

$$(2.03)^3 \approx 8.36.$$

Hint: Use $f(x) = x^3$ near $a = 2$.

6.

$$\sqrt[3]{26} \approx \frac{77}{27} \approx 2.852.$$

Hint: Use $f(x) = \sqrt[3]{x}$ near $a = 27$.

7.

$$\cos(0.08) \approx 1.$$

Hint: Use $f(x) = \cos x$ near $a = 0$.

8.

$$dy = (4x^3 + 6x)\, dx.$$

9.

$$dy = -\frac{1}{x^2}\, dx.$$

10.

$$dy = \frac{2}{\sqrt{4x+1}}\, dx.$$

11.

$$dy = 2x\cos(x^2)\, dx.$$

12.
$$|dV| \approx 180\pi \text{ cm}^3.$$

Hint:
$$V = \frac{4}{3}\pi r^3, \qquad dV = 4\pi r^2\, dr.$$

13.
$$|dV| \approx 9.6 \text{ cm}^3.$$

Hint:
$$V = s^3, \qquad dV = 3s^2\, ds.$$

14. Relative error:
$$\frac{1}{60}.$$

Percentage error:
$$\frac{5}{3}\% \approx 1.67\%.$$

Hint:
$$A = \pi r^2, \qquad dA = 2\pi r\, dr, \qquad \frac{|dA|}{A} \approx \frac{2|dr|}{r}.$$

15.
$$g(2.05) \approx -3.85,$$

$$g(1.98) \approx -4.06.$$

Hint:
$$L(x) = g(2) + g'(2)(x - 2).$$

Chapter 13: Maximum and Minimum Values

1. $x = 2$.
2. $x = -2, 2$.
3. $x = -2, 0, 2$.
4. $x = 0, \frac{1}{4}$.
5. $x = \frac{\pi}{4} + k\pi,\ k \in \mathbb{Z}$.
6. $t = 0$.

7. $x = -2, 0, 2.$

8. $x = 0, \frac{8}{7}, 4.$

9. $\theta = k\pi \pm \frac{\pi}{3},\ k \in \mathbb{Z}.$

10. $\theta = k\pi,\ k \in \mathbb{Z}.$

11. Absolute max: 6. Absolute min: -3.

12. Absolute max: 2. Absolute min: -2.

13. Absolute max: 45. Absolute min: -4.

14. Absolute max: 1. Absolute min: -1.

15. Absolute max: 3. Absolute min: 0.

16. Absolute max: 16. Absolute min: 7.

17. Absolute max: 113. Absolute min: 5.

18. Absolute max: 8. Absolute min: -19.

19. One possible graph passes through points like

$$(1,2),\ (2,0),\ (3,3),\ (4,1),\ (5,4).$$

20. One possible graph passes through points like

$$(1,1),\ (2,4),\ (3,0),\ (4,5),\ (5,-1).$$

21. Example:

$$f(x) = x, \qquad -1 \le x \le 2.$$

Absolute maximum at $x = 2$, no interior local maximum.

22. Example:

$$f(x) = \begin{cases} -x^2, & -1 \le x < 2, \\ 0, & x = 2. \end{cases}$$

23. Example:

$$f(x) = \begin{cases} x, & -1 < x < 2, \\ 0, & x = -1, \\ 2, & x = 2. \end{cases}$$

24. Example:

$$f(x) = \begin{cases} 2, & x = -1, \\ -1, & x = 0, \\ 0, & \text{otherwise.} \end{cases}$$

25. Absolute maximum value: 4. No absolute minimum.

26. Absolute maximum value: $\frac{8}{3}$. No absolute minimum.

27. Absolute minimum value: 0. No absolute maximum.

28. Absolute maximum value: 1. No absolute minimum.

29. Absolute minimum value: -1. No absolute maximum.

30. Absolute minimum value: -2. No absolute maximum.

31. Example:

$$f(x) = (x-1)^3.$$

Then $f'(1) = 0$, but $x = 1$ is not an extremum.

32. Example:

$$f(x) = |x|.$$

There is a local minimum at $x = 0$, but $f'(0)$ does not exist.

33. Continuity is necessary because a discontinuous function on a closed interval may fail to attain a maximum or minimum.

34. Yes. For example,

$$f(x) = x^2$$

on $[-1, 1]$ has absolute maximum value 1 at both $x = -1$ and $x = 1$.

35.

$$g'(x) = 3(x-5)^2.$$

So

$$g'(5) = 0.$$

Thus 5 is a critical number. But $g'(x) > 0$ for $x \neq 5$, so g is increasing on both sides of 5. Therefore, there is no local extremum at 5.

36.

$$f'(x) = 101x^{100} + 51x^{50} + 1.$$

Let $u = x^{50} \geq 0$. Then

$$f'(x) = 101u^2 + 51u + 1 > 0.$$

So f is always increasing and has neither a local maximum nor a local minimum.

Chapter 14: Derivatives and the Shape of a Graph

Derivative Graph Interpretation

1. From the graph,

$$f'(x) < 0 \text{ on } (0, 1.5) \cup (5, 6),$$

and

$$f'(x) > 0 \text{ on } (1.5, 5).$$

Therefore,

$$\boxed{f \text{ is increasing on } (1.5, 5)}$$

and

$$\boxed{f \text{ is decreasing on } (0, 1.5) \cup (5, 6).}$$

Since f' changes from negative to positive at $x = 1.5$,

$$\boxed{f \text{ has a local minimum at } x = 1.5.}$$

Since f' changes from positive to negative at $x = 5$,

$$\boxed{f \text{ has a local maximum at } x = 5.}$$

Hint: Use the sign of f'. A sign change from $-$ to $+$ gives a local minimum, while a change from $+$ to $-$ gives a local maximum.

Increasing/Decreasing, Extrema, Concavity, and Inflection Points

2. For

$$f(x) = x^3 - 3x^2 - 9x + 4,$$

$$f'(x) = 3(x+1)(x-3), \qquad f''(x) = 6(x-1).$$

Increasing:

$$\boxed{(-\infty, -1) \cup (3, \infty)}$$

Decreasing:

$$\boxed{(-1,3)}$$

Local maximum value:

$$\boxed{f(-1)=9}$$

Local minimum value:

$$\boxed{f(3)=-23}$$

Concave down:

$$\boxed{(-\infty,1)}$$

Concave up:

$$\boxed{(1,\infty)}$$

Inflection point:

$$\boxed{(1,-7)}$$

Hint: Use the signs of f' and f''.

3. For

$$f(x)=2x^3-9x^2+12x-3,$$

$$f'(x)=6(x-1)(x-2), \qquad f''(x)=12x-18.$$

Increasing:

$$\boxed{(-\infty,1)\cup(2,\infty)}$$

Decreasing:

$$\boxed{(1,2)}$$

Local maximum value:

$$\boxed{f(1)=2}$$

Local minimum value:

$$\boxed{f(2)=1}$$

Concave down:

$$\boxed{\left(-\infty,\frac{3}{2}\right)}$$

Concave up:

$$\boxed{\left(\frac{3}{2},\infty\right)}$$

Inflection point:

$$\boxed{\left(\frac{3}{2},\frac{3}{2}\right)}$$

4. For

$$f(x) = x^4 - 2x^2 + 3,$$

$$f'(x) = 4x(x-1)(x+1), \qquad f''(x) = 12x^2 - 4.$$

Increasing:

$$\boxed{(-1,0) \cup (1,\infty)}$$

Decreasing:

$$\boxed{(-\infty,-1) \cup (0,1)}$$

Local maximum value:

$$\boxed{f(0) = 3}$$

Local minimum value:

$$\boxed{f(-1) = f(1) = 2}$$

Concave up:

$$\boxed{\left(-\infty, -\frac{1}{\sqrt{3}}\right) \cup \left(\frac{1}{\sqrt{3}}, \infty\right)}$$

Concave down:

$$\boxed{\left(-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)}$$

Inflection points:

$$\boxed{\left(-\frac{1}{\sqrt{3}}, \frac{22}{9}\right), \quad \left(\frac{1}{\sqrt{3}}, \frac{22}{9}\right)}$$

5. For

$$f(x) = \frac{x}{x^2+1},$$

$$f'(x) = \frac{1-x^2}{(x^2+1)^2},$$

and

$$f''(x) = \frac{2x(x^2-3)}{(x^2+1)^3}.$$

Increasing:

$$\boxed{(-1,1)}$$

Decreasing:

$$\boxed{(-\infty,-1) \cup (1,\infty)}$$

Local minimum value:

$$\boxed{f(-1) = -\frac{1}{2}}$$

Local maximum value:

$$\boxed{f(1) = \frac{1}{2}}$$

Concave down:

$$\boxed{(-\infty, -\sqrt{3}) \cup (0, \sqrt{3})}$$

Concave up:

$$\boxed{(-\sqrt{3}, 0) \cup (\sqrt{3}, \infty)}$$

Inflection points:

$$\boxed{\left(-\sqrt{3}, -\frac{\sqrt{3}}{4}\right), \quad (0,0), \quad \left(\sqrt{3}, \frac{\sqrt{3}}{4}\right)}$$

6. For

$$f(x) = \sin x + \cos x, \qquad 0 \leq x \leq 2\pi,$$

$$f'(x) = \cos x - \sin x, \qquad f''(x) = -\sin x - \cos x.$$

Increasing:

$$\boxed{\left(0, \frac{\pi}{4}\right) \cup \left(\frac{5\pi}{4}, 2\pi\right)}$$

Decreasing:

$$\boxed{\left(\frac{\pi}{4}, \frac{5\pi}{4}\right)}$$

Local maximum value:

$$\boxed{f\left(\frac{\pi}{4}\right) = \sqrt{2}}$$

Local minimum value:

$$\boxed{f\left(\frac{5\pi}{4}\right) = -\sqrt{2}}$$

Concave down:

$$\boxed{\left(0, \frac{3\pi}{4}\right) \cup \left(\frac{7\pi}{4}, 2\pi\right)}$$

Concave up:

$$\boxed{\left(\frac{3\pi}{4}, \frac{7\pi}{4}\right)}$$

Inflection points:

$$\boxed{\left(\frac{3\pi}{4}, 0\right), \quad \left(\frac{7\pi}{4}, 0\right)}$$

7. For

$$f(x) = x^3 - 12x + 2,$$

$$f'(x) = 3(x-2)(x+2), \qquad f''(x) = 6x.$$

Increasing:

$$\boxed{(-\infty, -2) \cup (2, \infty)}$$

Decreasing:

$$\boxed{(-2, 2)}$$

Local maximum value:

$$\boxed{f(-2) = 18}$$

Local minimum value:

$$\boxed{f(2) = -14}$$

Concave down:

$$\boxed{(-\infty, 0)}$$

Concave up:

$$\boxed{(0, \infty)}$$

Inflection point:

$$\boxed{(0, 2)}$$

8. For

$$f(x) = 36x + 3x^2 - 2x^3,$$

$$f'(x) = -6(x-3)(x+2), \qquad f''(x) = 6 - 12x.$$

Increasing:

$$\boxed{(-2, 3)}$$

Decreasing:

$$\boxed{(-\infty, -2) \cup (3, \infty)}$$

Local minimum value:

$$\boxed{f(-2) = -44}$$

Local maximum value:

$$\boxed{f(3) = 81}$$

Concave up:

$$\boxed{\left(-\infty, \frac{1}{2}\right)}$$

Concave down:

$$\boxed{\left(\frac{1}{2}, \infty\right)}$$

Inflection point:

$$\boxed{\left(\frac{1}{2}, \frac{37}{2}\right)}$$

9. For

$$f(x) = \frac{1}{2}x^4 - 4x^2 + 3,$$

$$f'(x) = 2x(x-2)(x+2), \qquad f''(x) = 6x^2 - 8.$$

Increasing:

$$\boxed{(-2, 0) \cup (2, \infty)}$$

Decreasing:

$$\boxed{(-\infty, -2) \cup (0, 2)}$$

Local maximum value:

$$\boxed{f(0) = 3}$$

Local minimum value:

$$\boxed{f(-2) = f(2) = -5}$$

Concave up:

$$\boxed{\left(-\infty, -\frac{2\sqrt{3}}{3}\right) \cup \left(\frac{2\sqrt{3}}{3}, \infty\right)}$$

Concave down:

$$\boxed{\left(-\frac{2\sqrt{3}}{3}, \frac{2\sqrt{3}}{3}\right)}$$

Inflection points:

$$\boxed{\left(-\frac{2\sqrt{3}}{3}, -\frac{13}{9}\right), \quad \left(\frac{2\sqrt{3}}{3}, -\frac{13}{9}\right)}$$

Second Derivative Test

10. For

$$f(x) = x^4(x-1)^3,$$

$$f'(x) = x^3(x-1)^2(7x-4).$$

Therefore, the critical numbers are

$$\boxed{x = 0, \qquad x = \frac{4}{7}, \qquad x = 1.}$$

At $x = 0$,

$$f''(0) = 0,$$

so the Second Derivative Test is inconclusive. At $x = \frac{4}{7}$,

$$f''\left(\frac{4}{7}\right) > 0,$$

so f has a local minimum at

$$\boxed{x = \frac{4}{7}.}$$

At $x = 1$,

$$f''(1) = 0,$$

so the Second Derivative Test is inconclusive.

Using the First Derivative Test:

Interval	$f'(x)$
$(-\infty, 0)$	$+$
$\left(0, \frac{4}{7}\right)$	$-$
$\left(\frac{4}{7}, 1\right)$	$+$
$(1, \infty)$	$+$

Thus,

$$\boxed{x = 0 \text{ is a local maximum,}}$$

$$\boxed{x = \frac{4}{7} \text{ is a local minimum,}}$$

and

$$\boxed{x = 1 \text{ is neither a local maximum nor a local minimum.}}$$

Hint: The factor $(x-1)^2$ does not change sign at $x = 1$.

11. (a) Since

$$f'(2) = 0 \qquad \text{and} \qquad f''(2) = -5 < 0,$$

the Second Derivative Test shows that

$$\boxed{f \text{ has a local maximum at } x = 2.}$$

(b) Since

$$f'(6) = 0 \qquad \text{and} \qquad f''(6) = 0,$$

the Second Derivative Test is inconclusive.

$$\boxed{\text{No conclusion can be made without more information.}}$$

Curve Sketching

12. For

$$f(x) = x^3 - 12x + 2,$$

use the following key features:

$$\boxed{\text{Local maximum point } (-2, 18)}$$

$$\boxed{\text{Local minimum point } (2, -14)}$$

$$\boxed{\text{Inflection point } (0, 2)}$$

$$f(0) = 2.$$

The graph is increasing on

$$(-\infty, -2) \cup (2, \infty),$$

decreasing on

$$(-2, 2),$$

concave down for $x < 0$, and concave up for $x > 0$.

Hint: The left end goes downward and the right end goes upward.

13. For

$$f(x) = 36x + 3x^2 - 2x^3,$$

the key features are

$$\boxed{\text{Local minimum point } (-2, -44)}$$

$$\boxed{\text{Local maximum point } (3, 81)}$$

$$\boxed{\text{Inflection point } \left(\frac{1}{2}, \frac{37}{2}\right)}$$

and

$$f(0) = 0.$$

The graph is decreasing on

$$(-\infty, -2) \cup (3, \infty),$$

increasing on

$$(-2, 3),$$

concave up for $x < \frac{1}{2}$, and concave down for $x > \frac{1}{2}$.

Hint: Since the leading coefficient is negative, the left end goes upward and the right end goes downward.

14. For

$$f(x) = \frac{1}{2}x^4 - 4x^2 + 3,$$

the key features are

$$\boxed{\text{Local maximum point } (0, 3)}$$

and

$$\boxed{\text{Local minimum points } (-2, -5) \text{ and } (2, -5).}$$

The inflection points are

$$\boxed{\left(-\frac{2\sqrt{3}}{3}, -\frac{13}{9}\right), \quad \left(\frac{2\sqrt{3}}{3}, -\frac{13}{9}\right).}$$

The function is even, so the graph is symmetric about the y-axis.

The x-intercepts are

$$\boxed{x = \pm\sqrt{4 - \sqrt{10}}, \qquad x = \pm\sqrt{4 + \sqrt{10}}.}$$

Hint: Both ends of the graph rise toward $+\infty$.

Conceptual Problems

15. (a) A possible graph is an increasing, concave-down curve passing through

$$(3, 2) \qquad \text{with tangent slope} \qquad \frac{1}{2} \qquad \text{at} \qquad x = 3.$$

(b) The equation

$$f(x) = 0$$

has exactly

$$\boxed{1 \text{ solution.}}$$

Hint: Since $f'(x) > 0$, the function is strictly increasing, so it can have at most one zero. Since $f''(x) < 0$, f' is decreasing. For $x < 3$,

$$f'(x) > f'(3) = \frac{1}{2}.$$

This guarantees that $f(x)$ becomes negative sufficiently far to the left, while $f(3) = 2 > 0$.

(c) It is not possible that

$$f'(2) = \frac{1}{3}.$$

Since $f''(x) < 0$, the derivative f' is decreasing. Because $2 < 3$,

$$f'(2) > f'(3) = \frac{1}{2}.$$

Therefore,

$$\boxed{f'(2) \neq \frac{1}{3}.}$$

16. Suppose

$$f'(x) = (x+1)^2(x-3)^5(x-6)^4.$$

The even-power factors do not affect the sign. Thus, the sign is determined by

$$(x-3)^5.$$

Therefore,

$$f'(x) < 0 \quad \text{for } x < 3,$$

and

$$f'(x) > 0 \quad \text{for } x > 3.$$

Hence,

$$\boxed{f \text{ is increasing on } (3, \infty).}$$

Although $f'(6) = 0$, the derivative does not change sign there.

17. Let

$$h(x) = \tan x - x.$$

Then

$$h(0) = 0$$

and

$$h'(x) = \sec^2 x - 1 = \tan^2 x.$$

For

$$0 < x < \frac{\pi}{2},$$

we have

$$h'(x) > 0.$$

Therefore, h is increasing and

$$h(x) > h(0) = 0.$$

Thus,

$$\boxed{\tan x > x \quad \text{for} \quad 0 < x < \frac{\pi}{2}.}$$

18. Let

$$h(x) = 2\sqrt{x} - 3 + \frac{1}{x}.$$

Then

$$h(1) = 0.$$

Differentiate:

$$h'(x) = \frac{1}{\sqrt{x}} - \frac{1}{x^2}.$$

Rewrite:

$$h'(x) = \frac{x^{3/2} - 1}{x^2}.$$

For $x > 1$,

$$x^{3/2} - 1 > 0,$$

so

$$h'(x) > 0.$$

Thus h is increasing on $(1, \infty)$, and therefore

$$h(x) > h(1) = 0.$$

Hence,

$$\boxed{2\sqrt{x} > 3 - \frac{1}{x} \quad \text{for all } x > 1.}$$

Chapter 15: Limits at Infinity: Horizontal Asymptotes

Evaluate the Limits

1. $$\boxed{\frac{2}{5}}$$

2. $$\boxed{3}$$

   Hint: Divide the expression inside the square root by x^3.

3. $$\boxed{\frac{3}{2}}$$

4. $$\boxed{0}$$

5. $$\boxed{2}$$

6. $$\boxed{-1}$$

7. $$\boxed{2}$$

   Hint: Since $x \to -\infty$,

   $$\sqrt{x^6} = |x^3| = -x^3.$$

   Thus,

   $$\sqrt{1+4x^6} = (-x^3)\sqrt{4+\frac{1}{x^6}}.$$

8. $$\boxed{\frac{\sqrt{3}}{4}}$$

   Hint: Since $x \to \infty$,

$$\sqrt{x+3x^2} = x\sqrt{\frac{1}{x}+3}.$$

9.

$$\boxed{\frac{1}{6}}$$

Hint: Rationalize:

$$\sqrt{9x^2+x}-3x = \frac{x}{\sqrt{9x^2+x}+3x}.$$

10.

$$\boxed{-\frac{3}{4}}$$

Hint: Rationalize first. Also, because $x \to -\infty$,

$$\sqrt{x^2} = |x| = -x.$$

11.

$$\boxed{\frac{a-b}{2}}$$

Hint: Rationalize:

$$\sqrt{x^2+ax}-\sqrt{x^2+bx} = \frac{(a-b)x}{\sqrt{x^2+ax}+\sqrt{x^2+bx}}.$$

12.

$$\boxed{\text{The limit does not exist.}}$$

Hint: The function $\cos x$ continues to oscillate between -1 and 1.

13.

$$\boxed{\infty}$$

14.

$$\boxed{-\infty}$$

15.

$$\boxed{1}$$

Hint: Rewrite the expression as

$$x \sin\left(\frac{1}{x}\right) = \frac{\sin(1/x)}{1/x},$$

and use

$$\lim_{u\to 0} \frac{\sin u}{u} = 1.$$

16.

$$\boxed{0}$$

Hint: Rewrite:

$$\sqrt{x} \sin\left(\frac{1}{x}\right) = \frac{1}{\sqrt{x}} \cdot \frac{\sin(1/x)}{1/x}.$$

Horizontal Asymptotes

18.

$$\boxed{y = 4}$$

19.

$$\boxed{y = \frac{2}{3}}$$

20.

$$\boxed{y = 2}$$

21.

$$\boxed{y = -1}$$

22.

$$\boxed{\text{There is no horizontal asymptote.}}$$

Hint: The degree of the numerator is greater than the degree of the denominator.

End Behavior

23. The leading term is $-x^4$. Therefore,

$$\boxed{\lim_{x\to\infty} (2x^3 - x^4) = -\infty}$$

and

$$\boxed{\lim_{x\to-\infty} (2x^3 - x^4) = -\infty.}$$

Thus, both ends of the graph fall downward.

24. The leading term is $-x^6$. Therefore,

$$\boxed{\lim_{x\to\infty}(x^4 - x^6) = -\infty}$$

and

$$\boxed{\lim_{x\to-\infty}(x^4 - x^6) = -\infty.}$$

Thus, both ends of the graph fall downward.

25. The leading term is

$$x^3 \cdot x^2 \cdot x = x^6.$$

Therefore,

$$\boxed{\lim_{x\to\infty} x^3(x+2)^2(x-1) = \infty}$$

and

$$\boxed{\lim_{x\to-\infty} x^3(x+2)^2(x-1) = \infty.}$$

Thus, both ends of the graph rise upward.

26. The leading term is

$$(-x)(x^2)(x^4) = -x^7.$$

Therefore,

$$\boxed{\lim_{x\to\infty}(3-x)(1+x)^2(1-x)^4 = -\infty}$$

and

$$\boxed{\lim_{x\to-\infty}(3-x)(1+x)^2(1-x)^4 = \infty.}$$

Thus, the left end rises upward and the right end falls downward.

27. The leading term is

$$x^2 \cdot x^4 \cdot x = x^7.$$

Therefore,

$$\boxed{\lim_{x\to\infty} x^2(x^2-1)^2(x+2) = \infty}$$

and

$$\boxed{\lim_{x\to-\infty} x^2(x^2-1)^2(x+2) = -\infty.}$$

Thus, the left end falls downward and the right end rises upward.

Chapter 16: Curve Sketching

1.

$$f(x) = x^3 - 3x$$

$$f'(x) = 3(x-1)(x+1), \qquad f''(x) = 6x.$$

Increasing on:

$$\boxed{(-\infty, -1) \cup (1, \infty)}$$

Decreasing on:

$$\boxed{(-1, 1)}$$

Local maximum at:

$$\boxed{(-1, 2)}$$

Local minimum at:

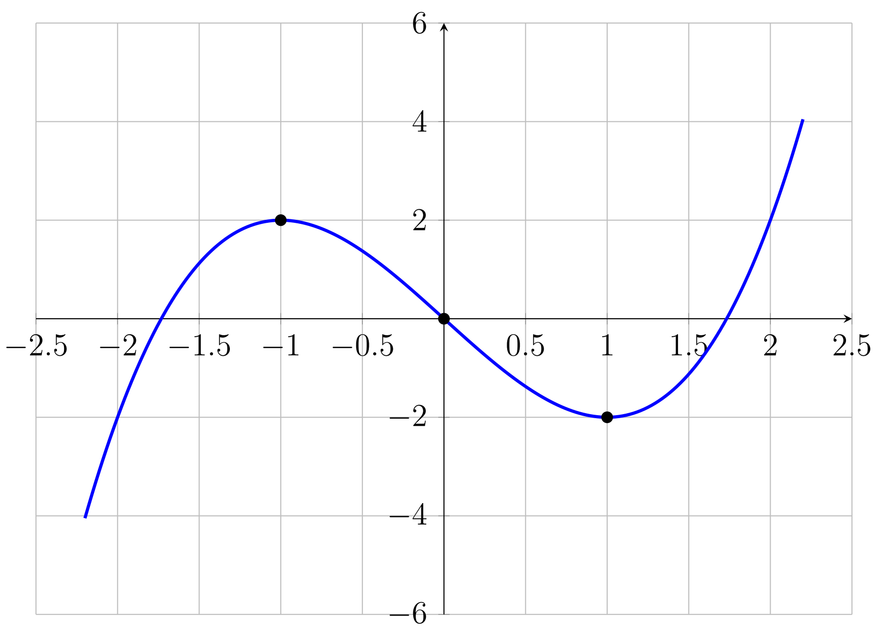
$$\boxed{(1, -2)}$$

Inflection point:

$$\boxed{(0, 0)}$$

Intercepts:

$$\boxed{(-\sqrt{3}, 0),\ (0, 0),\ (\sqrt{3}, 0)}$$



2.
$$f(x) = x^4 - 4x$$

$$f'(x) = 4(x-1)(x^2+x+1).$$

Since
$$x^2 + x + 1 > 0,$$

the only critical number is $x = 1$.

Decreasing on:
$$\boxed{(-\infty, 1)}$$

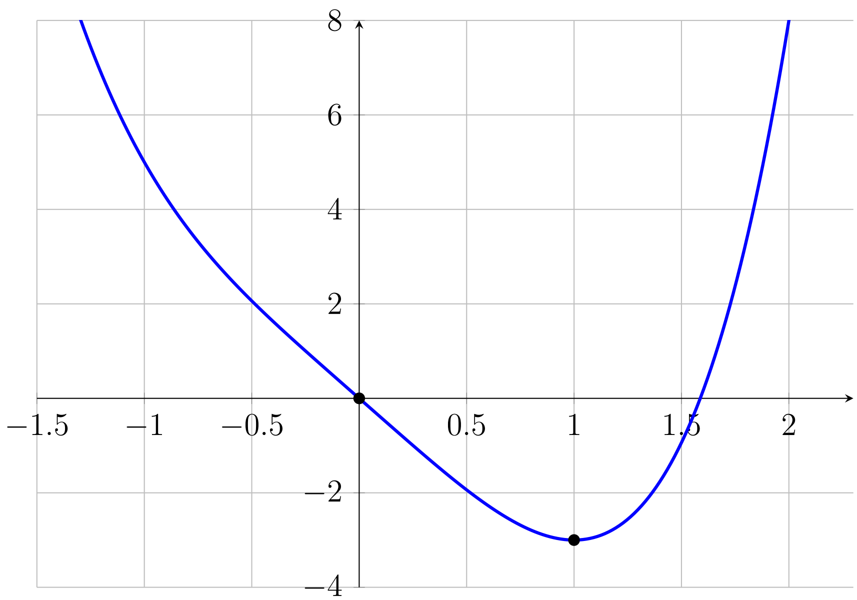
Increasing on:
$$\boxed{(1, \infty)}$$

Absolute and local minimum at:
$$\boxed{(1, -3)}$$

Concavity at:
$$\boxed{\text{Concave up on } (-\infty, \infty)}$$

There are no inflection points.

Intercepts at:
$$\boxed{(0,0), \quad \left(\sqrt[3]{4}, 0\right)}$$



3.
$$f(x) = 2 + 3x^2 - x^3$$

$$f'(x) = -3x(x-2), \qquad f''(x) = -6(x-1).$$

Increasing:

$$\boxed{(0,2)}$$

Decreasing:

$$\boxed{(-\infty,0)\cup(2,\infty)}$$

Local minimum:

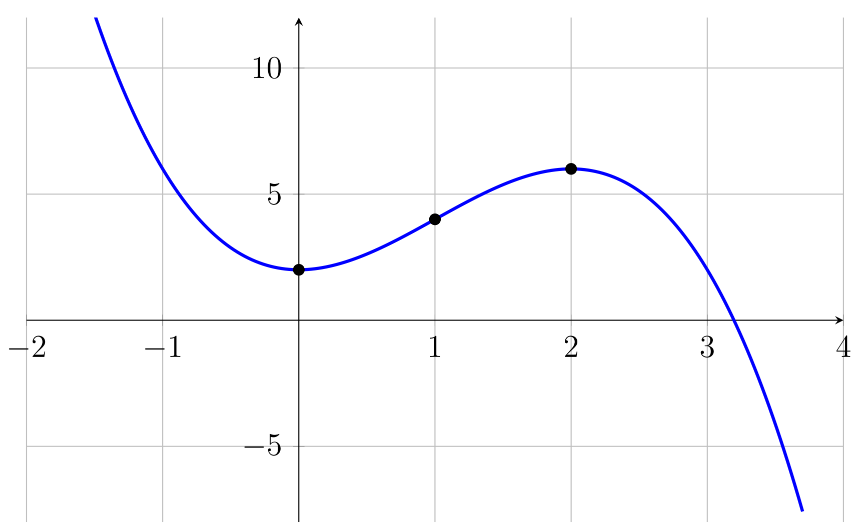
$$\boxed{(0,2)}$$

Local maximum:

$$\boxed{(2,6)}$$

Inflection point:

$$\boxed{(1,4)}$$



4.
$$f(x) = \frac{x^2 + 5x}{25 - x^2}$$

Factor and simplify:

$$f(x) = \frac{x(x+5)}{(5-x)(5+x)} = \frac{x}{5-x}, \qquad x \neq -5, 5.$$

Vertical asymptote:

$$\boxed{x = 5}$$

Horizontal asymptote:

$$\boxed{y = -1}$$

Hole at:

$$\boxed{\left(-5, -\frac{1}{2}\right)}$$

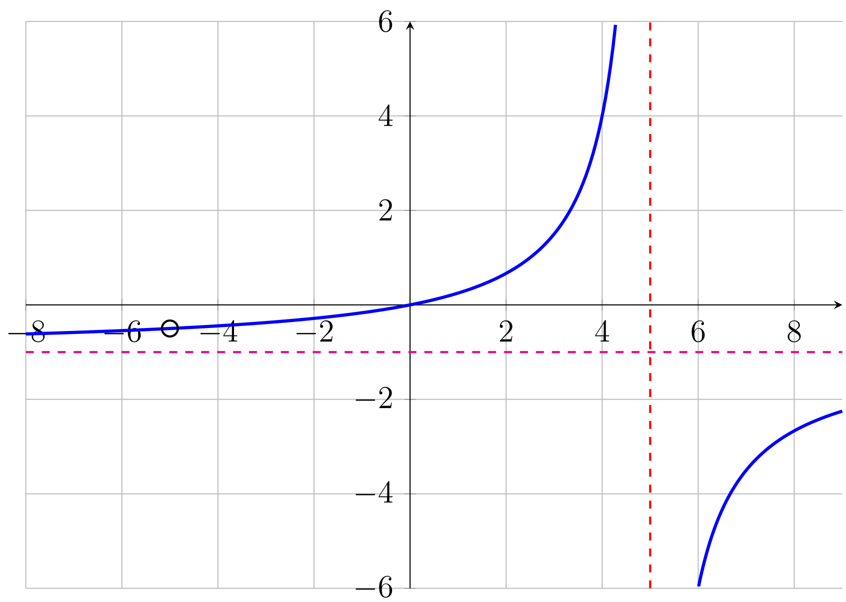
x and y intercepts pf this function is the same. It is at:

$$\boxed{(0, 0)}$$

The function is increasing on every interval of its domain:

$$\boxed{(-\infty, -5), \quad (-5, 5), \quad (5, \infty)}$$

Note: Do not forget that cancellation creates a hole at $x = -5$, not an ordinary point.



5.
$$f(x) = \frac{x - x^2}{2 - 3x + x^2}$$

Factor:
$$f(x) = \frac{-x(x-1)}{(x-1)(x-2)} = -\frac{x}{x-2}, \qquad x \neq 1, 2.$$

Vertical asymptote:
$$\boxed{x = 2}$$

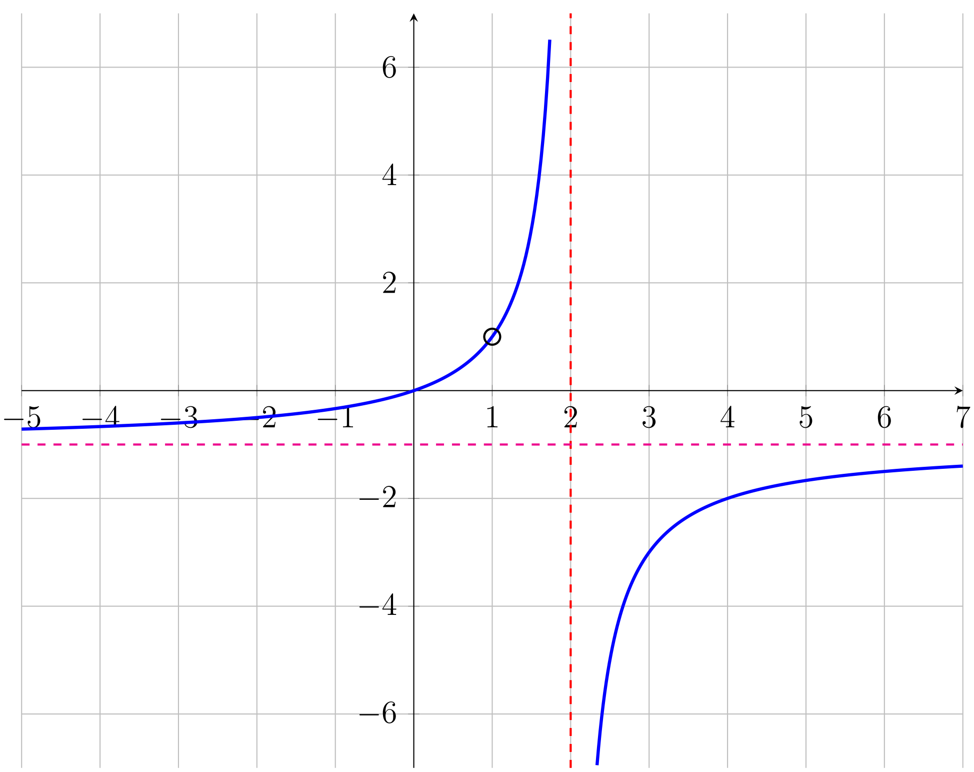
Horizontal asymptote:
$$\boxed{y = -1}$$

Hole at:
$$\boxed{(1, 1)}$$

x and y intercepts are the same at:
$$\boxed{(0, 0)}$$

Increasing on:
$$\boxed{(-\infty, 1), \quad (1, 2), \quad (2, \infty)}$$

Note: The canceled factor $x - 1$ produces a removable discontinuity.



6.
$$f(x) = 1 + \frac{1}{x} + \frac{1}{x^2}$$

Vertical asymptote:
$$\boxed{x = 0}$$

Horizontal asymptote:
$$\boxed{y = 1}$$

Critical number:
$$\boxed{x = -2}$$

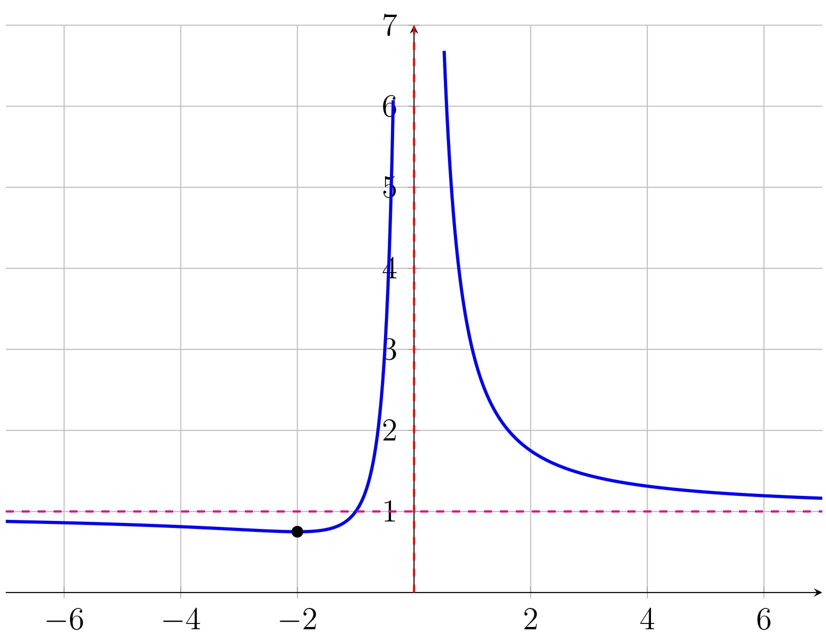
Local minimum at:
$$\boxed{\left(-2, \frac{3}{4}\right)}$$

Decreasing on:
$$\boxed{(-\infty, -2) \cup (0, \infty)}$$

Increasing on:
$$\boxed{(-2, 0)}$$

Inflection point:
$$\boxed{\left(-3, \frac{7}{9}\right)}$$

There are no real x-intercepts because $x^2 + x + 1 = 0$ has no real solutions.



7.
$$f(x) = \frac{x}{x^2 - 4}$$

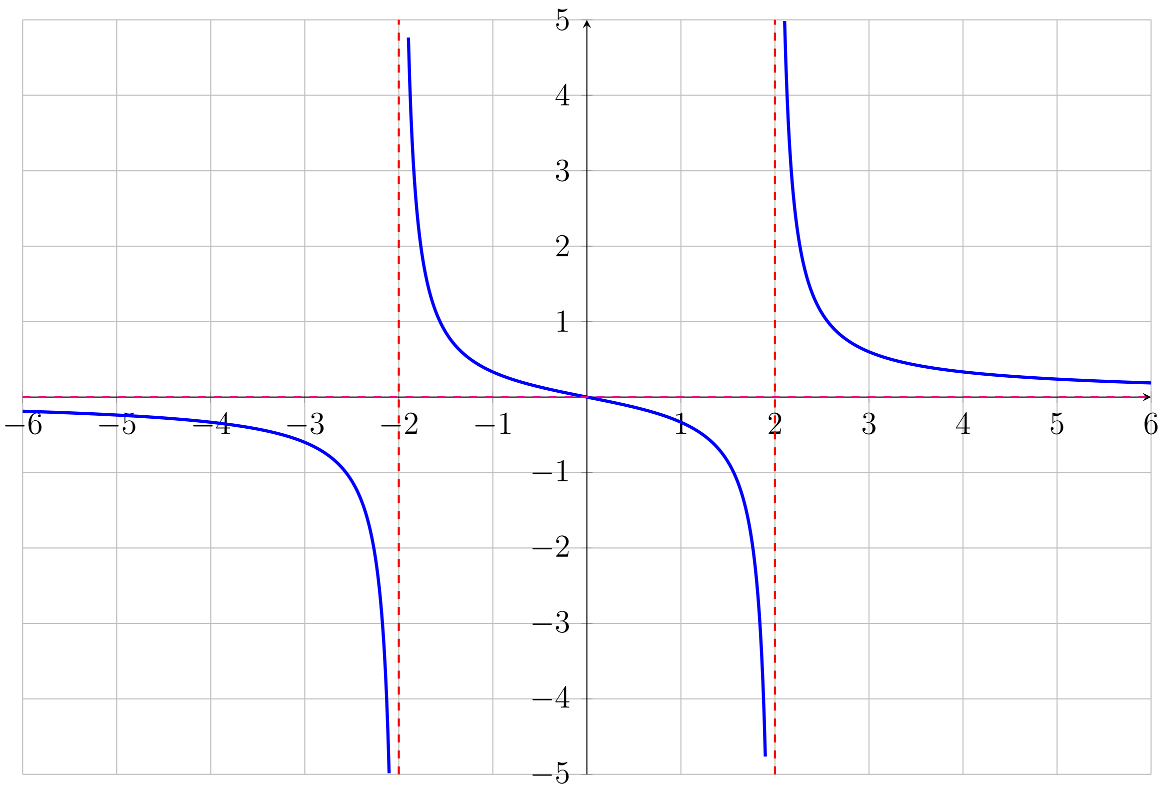
Vertical asymptotes:
$$\boxed{x = -2, \qquad x = 2}$$

Horizontal asymptote:
$$\boxed{y = 0}$$

Intercept:
$$\boxed{(0, 0)}$$

Since
$$f'(x) = -\frac{x^2 + 4}{(x^2 - 4)^2} < 0,$$
the function is decreasing on every interval of its domain:
$$\boxed{(-\infty, -2), \quad (-2, 2), \quad (2, \infty)}$$

Inflection point:
$$\boxed{(0, 0)}$$



8.
$$f(x) = \frac{1}{x^2 - 4}$$

Vertical asymptotes:
$$\boxed{x = -2, \qquad x = 2}$$

Horizontal asymptote:
$$\boxed{y = 0}$$

Local maximum:
$$\boxed{\left(0, -\frac{1}{4}\right)}$$

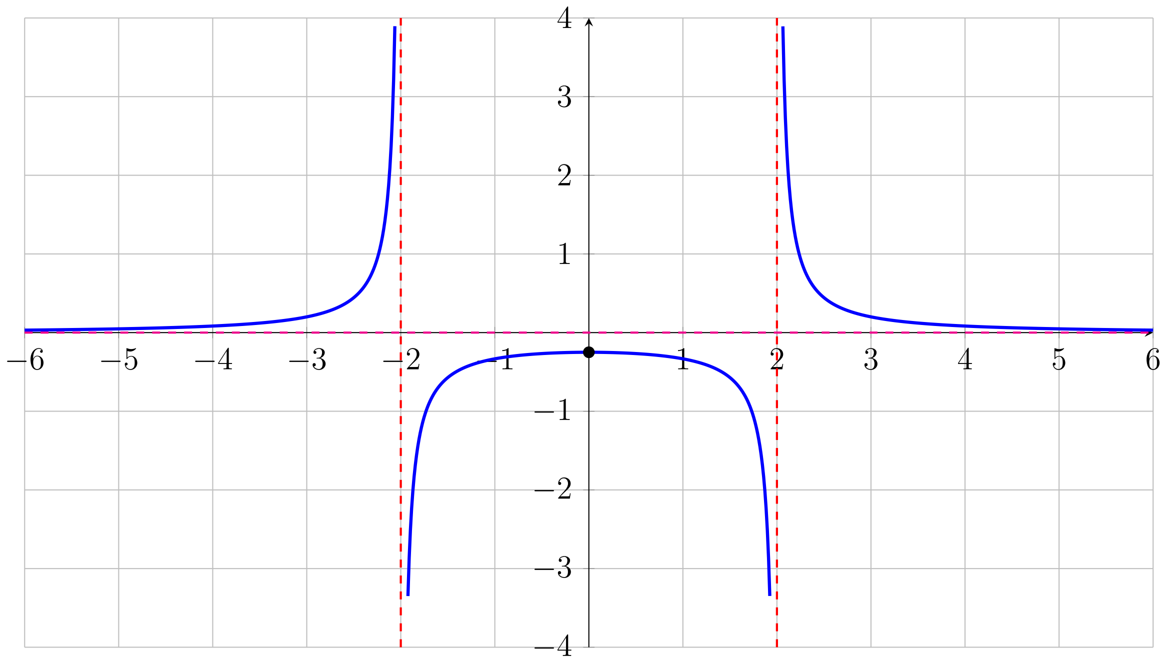
Increasing:
$$\boxed{(-\infty, -2) \cup (-2, 0)}$$

Decreasing:
$$\boxed{(0, 2) \cup (2, \infty)}$$

Concave up:
$$\boxed{(-\infty, -2) \cup (2, \infty)}$$

Concave down:
$$\boxed{(-2, 2)}$$

There are no inflection points because the possible changes at $x = \pm 2$ occur where the function is not defined.



9.

$$f(x) = x + \cos x$$

$$f'(x) = 1 - \sin x \geq 0.$$

Therefore,

$$\boxed{f \text{ is increasing on } (-\infty, \infty).}$$

There are horizontal tangents at

$$\boxed{x = \frac{\pi}{2} + 2k\pi, \qquad k \in \mathbb{Z},}$$

but there are no local extrema because f' does not change sign.

Inflection points occur where

$$\cos x = 0.$$

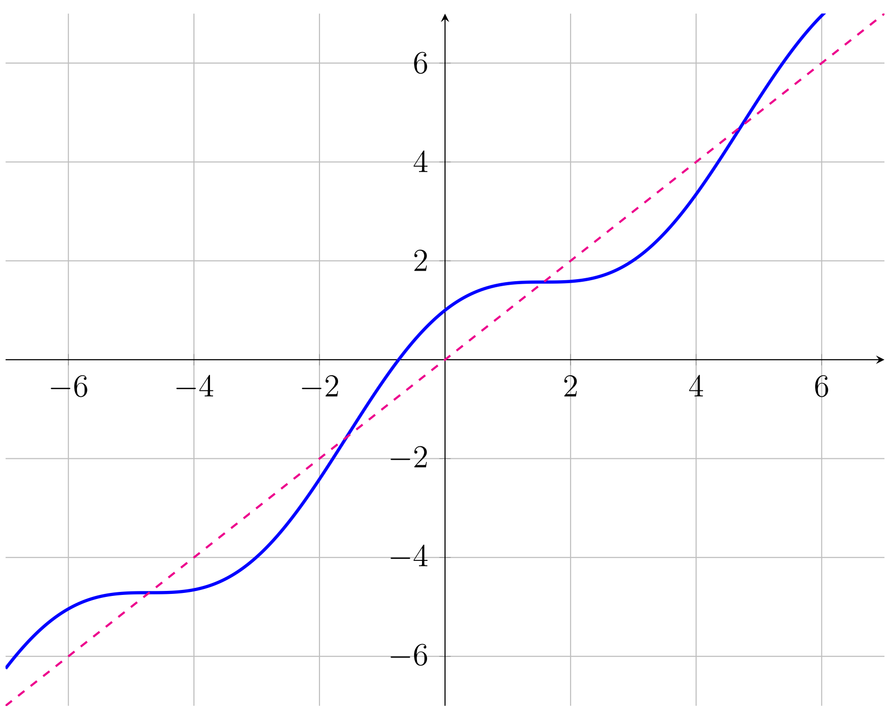
Thus,

$$\boxed{x = \frac{\pi}{2} + k\pi, \qquad k \in \mathbb{Z}.}$$

The graph oscillates around the line

$$y = x,$$

but $y = x$ is not an asymptote because the difference $\cos x$ does not approach 0.



10.

$$f(x) = x \tan x, \qquad -\frac{\pi}{2} < x < \frac{\pi}{2}.$$

The function is an even function.

Vertical asymptotes are:

$$\boxed{x = -\frac{\pi}{2}, \qquad x = \frac{\pi}{2}}$$

Decreasing on:

$$\boxed{\left(-\frac{\pi}{2}, 0\right)}$$

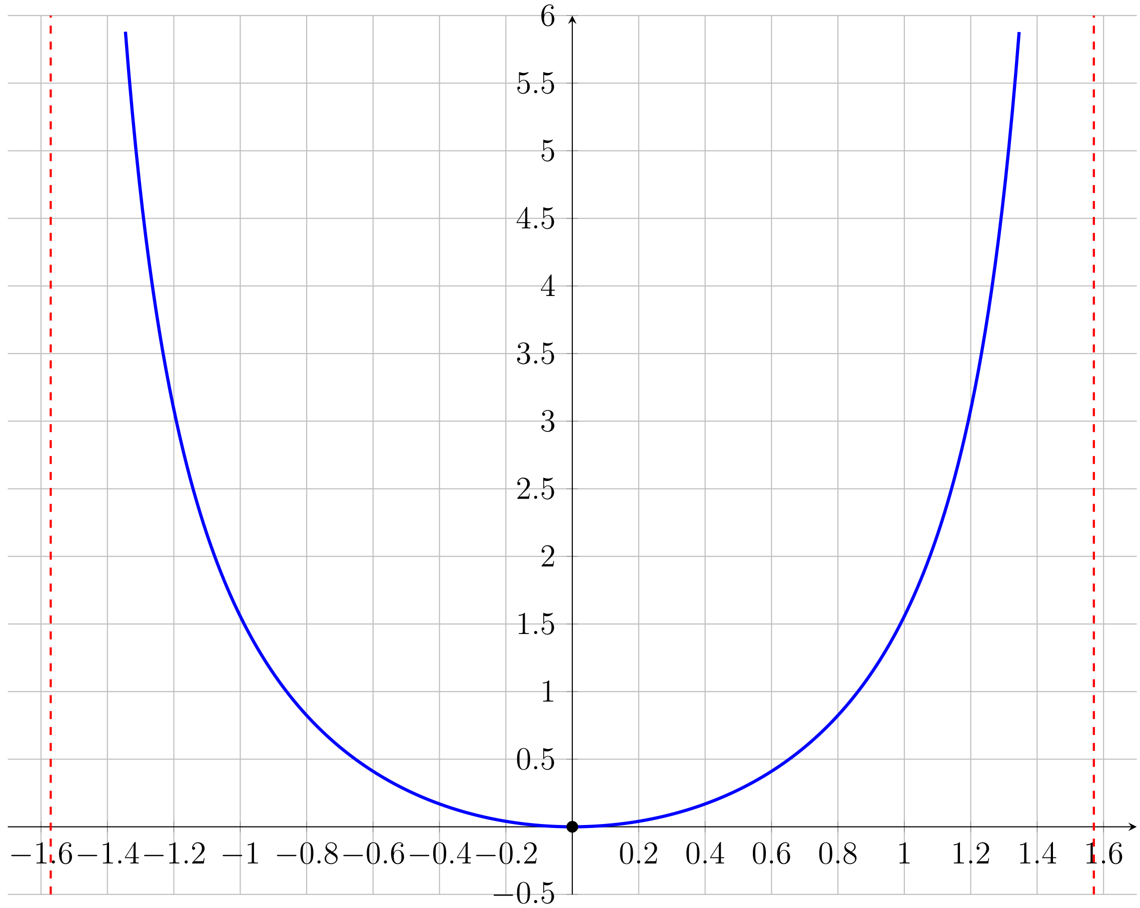
Increasing on:

$$\boxed{\left(0, \frac{\pi}{2}\right)}$$

Absolute and local minimum at:

$$\boxed{(0, 0)}$$

The function is concave up throughout its domain.



11.
$$f(x) = 2x - \tan x, \qquad -\frac{\pi}{2} < x < \frac{\pi}{2}.$$

Vertical asymptotes are:
$$\boxed{x = -\frac{\pi}{2}, \qquad x = \frac{\pi}{2}}$$

Critical numbers:
$$\boxed{x = -\frac{\pi}{4}, \qquad x = \frac{\pi}{4}}$$

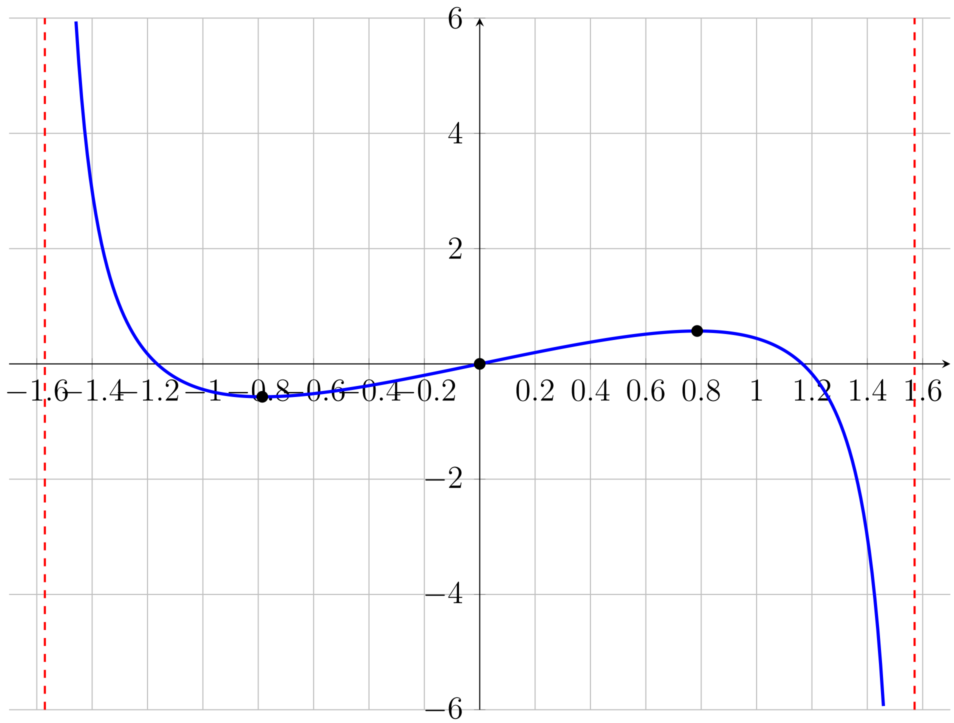
Local minimum at:
$$\boxed{\left(-\frac{\pi}{4}, 1 - \frac{\pi}{2}\right)}$$

Local maximum at:
$$\boxed{\left(\frac{\pi}{4}, \frac{\pi}{2} - 1\right)}$$

Decreasing on:
$$\boxed{\left(-\frac{\pi}{2}, -\frac{\pi}{4}\right) \cup \left(\frac{\pi}{4}, \frac{\pi}{2}\right)}$$

Increasing on:
$$\boxed{\left(-\frac{\pi}{4}, \frac{\pi}{4}\right)}$$

Inflection point:
$$\boxed{(0, 0)}$$



12.
$$f(x) = \frac{\sin x}{1 + \cos x}.$$

Using a half-angle identity,
$$f(x) = \tan\left(\frac{x}{2}\right).$$

Domain:
$$\boxed{x \neq (2k+1)\pi, \qquad k \in \mathbb{Z}}$$

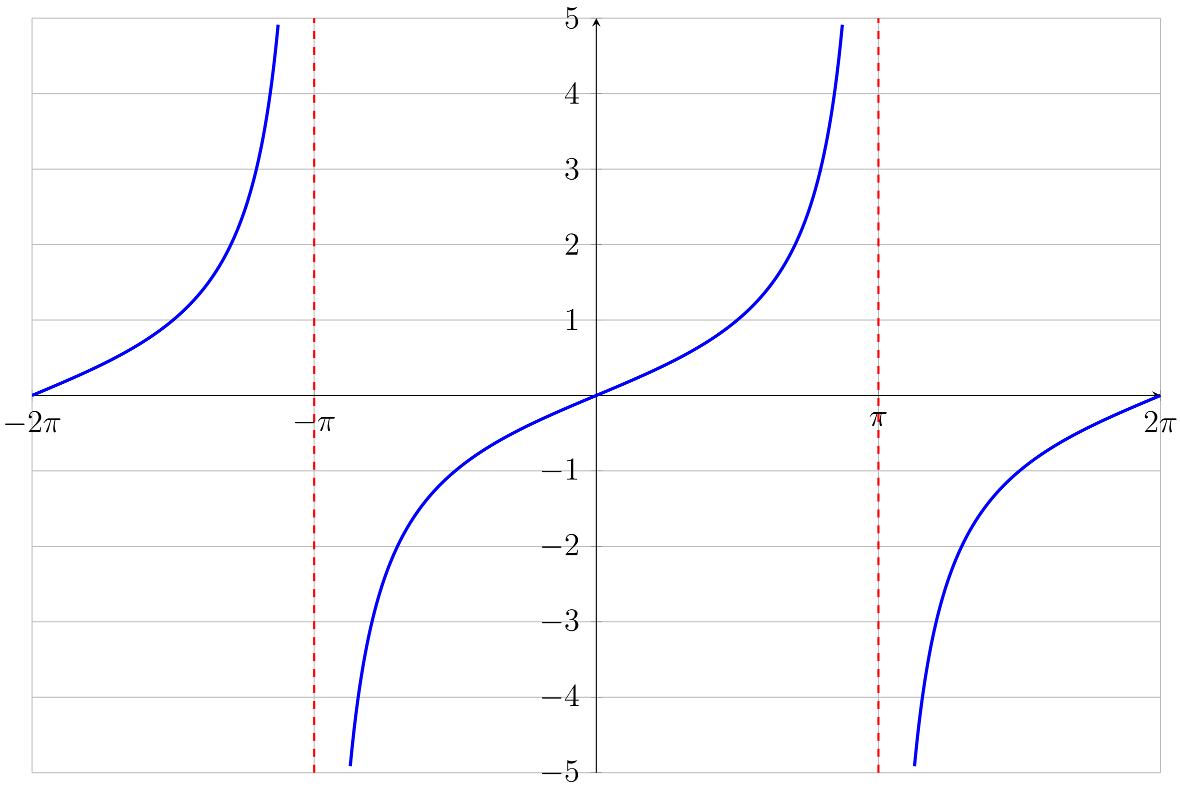
Vertical asymptotes:
$$\boxed{x = (2k+1)\pi}$$

Zeros:
$$\boxed{x = 2k\pi}$$

The function is increasing on every interval
$$\boxed{\big((2k-1)\pi, (2k+1)\pi\big)}.$$

Inflection points:
$$\boxed{(2k\pi, 0)}$$

Hint: Rewrite the function as $\tan(x/2)$ before sketching.



13.
$$f(x) = \frac{4x^3 - 10x^2 - 11x + 1}{x^2 - 3x}.$$

Polynomial division gives

$$f(x) = 4x + 2 + \frac{-5x + 1}{x^2 - 3x}.$$

Therefore, the slant asymptote is

$$\boxed{y = 4x + 2.}$$

Hint: Divide the numerator by the denominator and discard the remainder term when finding the asymptote.

14.
$$f(x) = \frac{2x^3 - 5x^2 + 3x}{x^2 - x - 2}.$$

Polynomial division gives

$$f(x) = 2x - 3 + \frac{4x - 6}{x^2 - x - 2}.$$

Therefore, the slant asymptote is

$$\boxed{y = 2x - 3.}$$

15.
$$f(x) = \sqrt{x^2 + 4x}.$$

The domain is determined by
$$x(x + 4) \geq 0,$$

so
$$\boxed{\mathrm{Dom}(f) = (-\infty, -4] \cup [0, \infty).}$$

As $x \to \infty$,
$$\sqrt{x^2 + 4x} - (x + 2) \to 0,$$

so the right-hand slant asymptote is

$$\boxed{y = x + 2.}$$

As $x \to -\infty$,
$$\sqrt{x^2 + 4x} - (-x - 2) \to 0,$$

so the left-hand slant asymptote is

$$\boxed{y = -x - 2.}$$

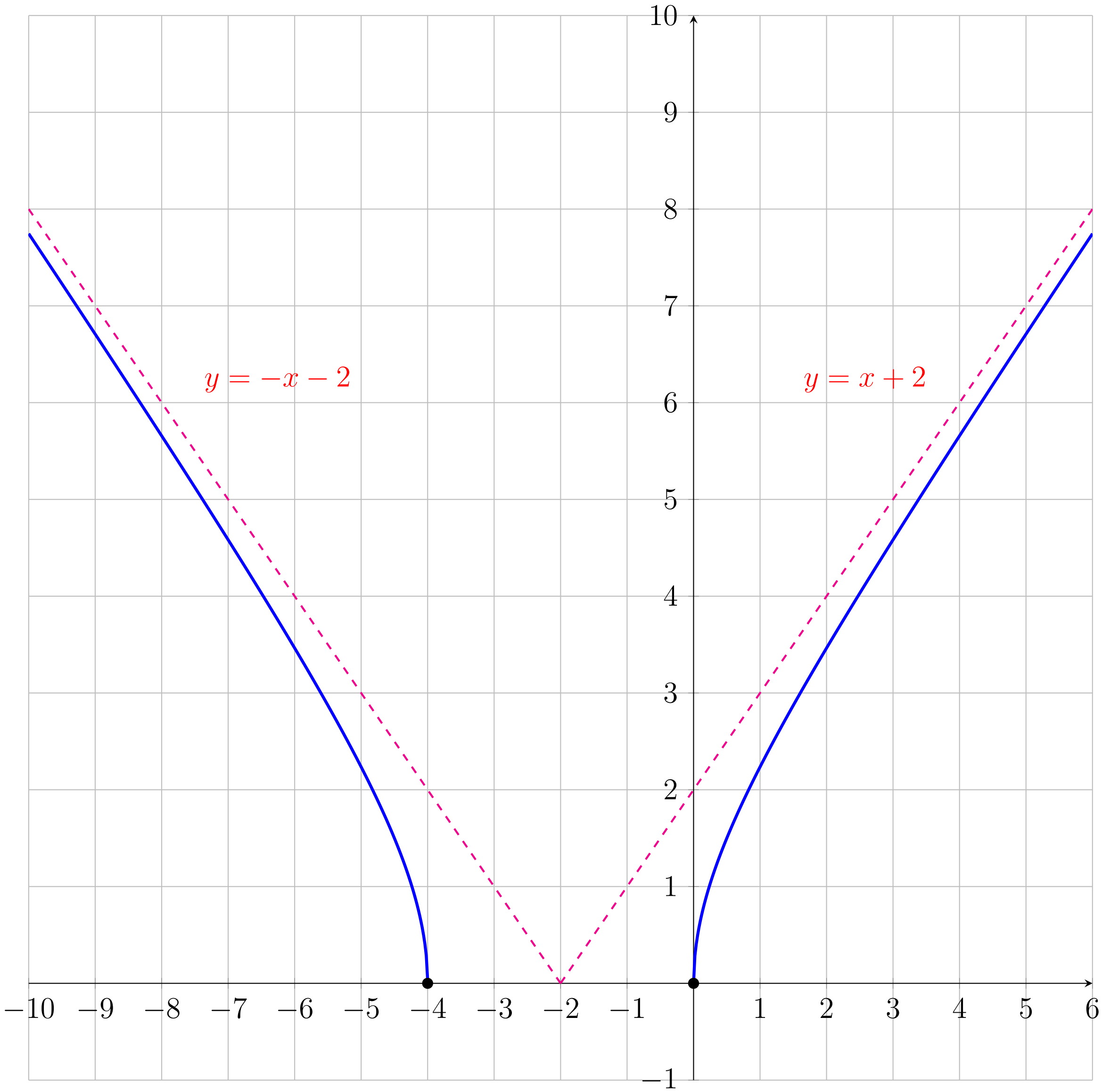
The graph passes through

$$\boxed{(-4, 0) \quad \text{and} \quad (0, 0).}$$

Both branches lie below their corresponding slant asymptotes.

Hint: For $x \to -\infty$, remember that

$$\sqrt{x^2} = |x| = -x.$$



Chapter 17: Optimization Problems

1. The two numbers are

$$\boxed{15 \text{ and } 15}.$$

   The maximum product is

$$\boxed{225}.$$

2. The two numbers are

$$\boxed{12 \text{ and } 12}.$$

   The minimum sum is

$$\boxed{24}.$$

   Hint: Let the numbers be x and $\dfrac{144}{x}$, and minimize

$$S(x) = x + \frac{144}{x}.$$

3. The dimensions are

$$\boxed{20 \text{ m} \times 20 \text{ m}}.$$

   The maximum area is

$$\boxed{400 \text{ m}^2}.$$

4. The dimensions are

$$\boxed{10\sqrt{2} \text{ m} \times 10\sqrt{2} \text{ m}}.$$

   The minimum perimeter is

$$\boxed{40\sqrt{2} \text{ m}}.$$

5. The dimensions are

$$\boxed{100 \text{ ft} \times 100 \text{ ft}}.$$

   The maximum enclosed area is

$$\boxed{10{,}000 \text{ ft}^2}.$$

6. The dimensions are

$$\boxed{20\sqrt{3} \text{ ft} \times 20\sqrt{3} \text{ ft}}.$$

   The minimum amount of fencing is

$$\boxed{80\sqrt{3} \text{ ft}}.$$

7. The box dimensions are

$$\boxed{8 \text{ in} \times 8 \text{ in} \times 2 \text{ in}}.$$

The maximum volume is

$$\boxed{128 \text{ in}^3}.$$

Hint: The volume function is

$$V(x) = x(12 - 2x)^2, \qquad 0 < x < 6.$$

8. The box dimensions are

$$\boxed{12 \text{ in} \times 6 \text{ in} \times 2 \text{ in}}.$$

The maximum volume is

$$\boxed{144 \text{ in}^3}.$$

Hint: The volume function is

$$V(x) = x(16 - 2x)(10 - 2x), \qquad 0 < x < 5.$$

The only critical number in the physical domain that gives a maximum is

$$x = 2.$$

9. The volume function is

$$\boxed{V(x) = x(20 - 2x)^2}.$$

The physical domain is

$$\boxed{0 < x < 10}.$$

If degenerate boxes are included, the domain can be written as

$$\boxed{0 \le x \le 10}.$$

10. Let x be the side length of the square base. The surface area is

$$\boxed{S(x) = x^2 + \frac{2000}{x}}, \qquad x > 0.$$

Hint: If h is the height, then

$$x^2h = 500.$$

Thus,

$$h = \frac{500}{x^2}.$$

Since the box has no top,

$$S = x^2 + 4xh.$$

11. The area function is

$$\boxed{A(x) = 2x(12 - x^2)}.$$

Its physical domain is

$$\boxed{0 \le x \le 2\sqrt{3}}.$$

Hint: The rectangle has width $2x$ and height

$$12 - x^2.$$

12. The area function is

$$\boxed{A(x) = 2x\sqrt{25 - x^2}}.$$

Its physical domain is

$$\boxed{0 \le x \le 5}.$$

Hint: From the semicircle,

$$y = \sqrt{25 - x^2}.$$

The rectangle has width $2x$ and height y.

Chapter 18: Antiderivatives

Practice Problems A

1.

$$\boxed{F(x) = \frac{x^3}{3} - \frac{3x^2}{2} + 2x + C}$$

2.

$$\boxed{F(x) = x^6 - \frac{8}{5}x^5 - 3x^3 + C}$$

3.

$$\boxed{F(x) = \sqrt{2}\,x + C}$$

4.
$$\boxed{F(x) = \pi^2 x + C}$$

5.
$$\boxed{F(x) = -\frac{5}{4x^8} + C}$$

Hint: Rewrite
$$\frac{10}{x^9} = 10x^{-9}$$
and use the Power Rule.

6.
$$\boxed{F(x) = -\frac{1}{x^5} + \frac{2}{x^2} + 2x + C}$$

Hint: First rewrite the function as
$$\frac{5 - 4x^3 + 2x^6}{x^6} = 5x^{-6} - 4x^{-3} + 2.$$

7.
$$\boxed{G(t) = 2\sqrt{t} + \frac{2}{3}t^{3/2} + \frac{2}{5}t^{5/2} + C}$$

Hint: Rewrite:
$$\frac{1 + t + t^2}{\sqrt{t}} = t^{-1/2} + t^{1/2} + t^{3/2}.$$

8.
$$\boxed{H(\theta) = -2\cos\theta - \tan\theta + C}$$

9.
$$\boxed{G(v) = 5v + 3\tan v + C}$$

10.
$$\boxed{F(t) = \frac{16}{3}t^{3/2} - \sec t + C}$$

Hint: Recall that
$$\frac{d}{dt}(\sec t) = \sec t \tan t.$$

11.
$$\boxed{F(x) = x - 2\cos x + 6\sqrt{x} + C}$$

12.
$$\boxed{F(x) = 4x^3 + 4x^2 + C}$$

Hint: Expand first:

$$x(12x + 8) = 12x^2 + 8x.$$

13.

$$\boxed{F(x) = 4x - \frac{3}{4}x^{4/3} + C}$$

14.

$$\boxed{F(x) = \frac{3}{5}x^{5/3} + 3x^{1/3} + C}$$

15.

$$\boxed{F(t) = 12t - \cos t + C}$$

Practice Problems B

1.

$$\boxed{f(x) = x^5 - x^4 + x^3 + C_1x + C_2}$$

Hint: Integrate twice. Two arbitrary constants are needed.

2.

$$\boxed{f(x) = \frac{x^8}{56} - \frac{2x^6}{15} + \frac{x^3}{6} + \frac{x^2}{2} + C_1x + C_2}$$

3.

$$\boxed{f(x) = 2x^2 - \frac{9}{28}x^{7/3} + C_1x + C_2}$$

4.

$$\boxed{f(x) = \frac{9}{40}x^{8/3} + \frac{9}{4}x^{4/3} + C_1x + C_2}$$

Hint: Rewrite the powers using exponents and integrate twice. Since $x^{-2/3}$ is undefined at $x = 0$, the formula applies on intervals not containing 0.

5.

$$\boxed{f(t) = 2t^3 + \cos t + C_1t^2 + C_2t + C_3}$$

Hint: Since f''' is given, integrate three times.

6.

$$\boxed{f(x) = x + 2x^{3/2} + 5}$$

7.
$$\boxed{f(x) = x^5 - x^3 + 4x + 6}$$

8.
$$\boxed{f(x) = 4x^{3/2} + 2x^{5/2} + 4}$$

Hint: Rewrite
$$\sqrt{x}(6 + 5x) = 6x^{1/2} + 5x^{3/2}.$$

9.
$$\boxed{f(t) = \frac{t^2}{2} - \frac{1}{2t^2} + 6}$$

10.
$$\boxed{f(t) = \tan t + \sec t - 2 - \sqrt{2}}$$

Hint: Split the derivative as
$$f'(t) = \sec^2 t + \sec t \tan t.$$

11.
$$\boxed{f(x) = \frac{2}{3}x^{3/2} + 2\sqrt{x} + \frac{7}{3}}$$

12.
$$\boxed{f(x) = -x^2 + 2x^3 - x^4 + 12x + 4}$$

13.
$$\boxed{f(x) = \frac{2}{5}x^5 + \frac{5}{2}x^2 + x - \frac{39}{10}}$$

14.
$$\boxed{f(\theta) = -\sin\theta - \cos\theta + 5\theta + 4}$$

15.
$$\boxed{f(t) = 2t^2 - \frac{1}{t^2} + \frac{3}{4}t + \frac{17}{4}}$$

Hint: First integrate f'' and use $f'(2) = 9$ to find the first constant. Then integrate again and use $f(1) = 6$.

16.
$$\boxed{f(x) = 2x^2 + x^3 + 2x^4 + 2x + 3}$$

Hint: After integrating twice, use both values $f(0) = 3$ and $f(1) = 10$ to determine the two constants.

17.

$$\boxed{f(x) = x^5 + x^4 + 2x^2 - 7x + 8}$$

18.

$$\boxed{f(1) = 8}$$

Indeed,

$$f(x) = 3x - 2x^2 + 7.$$

19.

$$\boxed{f(x) = \frac{x^4}{4} + \frac{3}{4}}$$

Hint: Since the tangent line is

$$y = -x,$$

its slope is -1. If the point of tangency occurs at $x = a$, then

$$f'(a) = a^3 = -1,$$

so $a = -1$. The point of tangency lies on $y = -x$, so

$$f(-1) = 1.$$

Practice Problems C

1.

$$\boxed{s(t) = 1 - \cos t - \sin t}$$

2.

$$\boxed{s(t) = \frac{t^3}{3} - 2t^{3/2} + \frac{8}{3}}$$

3.

$$\boxed{s(t) = \frac{t^3}{3} + \frac{t^2}{2} - 2t + 3}$$

Hint: Integrate the acceleration to find $v(t)$, use $v(0) = -2$, and then integrate again to find $s(t)$.

4.
$$\boxed{s(t) = -3\cos t + 2\sin t + 2t + 3}$$

5.
$$\boxed{s(t) = -10\sin t - 3\cos t + \frac{6}{\pi}t + 3}$$

Hint: Integrating the acceleration twice gives

$$s(t) = -10\sin t - 3\cos t + C_1 t + C_2.$$

Use both position conditions to determine C_1 and C_2.

6.
$$\boxed{s(t) = \frac{t^4}{12} - \frac{2}{3}t^3 + 3t^2 + \frac{211}{12}t}$$

Hint: After integrating twice, use $s(0) = 0$ and $s(1) = 20$.

For Problems 7–10, take upward as positive and use

$$a(t) = -10 \text{ m/s}^2.$$

7.
$$\boxed{s(t) = 450 - 5t^2}$$

8. The stone reaches the ground when $s(t) = 0$:

$$450 - 5t^2 = 0.$$

Thus,

$$\boxed{t = 3\sqrt{10} \text{ s} \approx 9.49 \text{ s.}}$$

9. The velocity is

$$v(t) = -10t.$$

At impact,

$$\boxed{v = -30\sqrt{10} \text{ m/s} \approx -94.9 \text{ m/s.}}$$

Thus, the impact speed is

$$\boxed{30\sqrt{10} \text{ m/s} \approx 94.9 \text{ m/s.}}$$

10. When the stone is thrown downward at 5 m/s,

$$s(t) = 450 - 5t - 5t^2.$$

Solving $s(t) = 0$ gives

$$\boxed{t = 9 \text{ s.}}$$

Additional Applications

For problems involving feet, use

$$g = 32 \text{ ft/s}^2.$$

11. Let $t = 0$ be the time at which the first ball is thrown. Their positions relative to the top of the cliff are

$$s_1(t) = 48t - 16t^2$$

and, for $t \geq 1$,

$$s_2(t) = 24(t-1) - 16(t-1)^2.$$

Setting $s_1(t) = s_2(t)$ gives

$$\boxed{t = 5 \text{ s.}}$$

Thus, their paths meet 5 seconds after the first ball is thrown, or 4 seconds after the second ball is thrown.

The meeting position is

$$s_1(5) = -160 \text{ ft.}$$

Therefore, they pass each other while in the air only if the cliff is more than 160 ft high.

$$\boxed{\text{They meet 160 ft below the launch point.}}$$

12. Since the stone is dropped,

$$v^2 = 2gh.$$

Thus,

$$120^2 = 2(32)h,$$

so

$$\boxed{h = 225 \text{ ft.}}$$

13. Convert the change in speed to feet per second:

$$50 - 30 = 20 \text{ mi/h} = 20\left(\frac{5280}{3600}\right) = \frac{88}{3} \text{ ft/s.}$$

Therefore,

$$a = \frac{\Delta v}{\Delta t} = \frac{88/3}{5}.$$

Hence,

$$\boxed{a = \frac{88}{15} \text{ ft/s}^2 \approx 5.87 \text{ ft/s}^2.}$$

14. Using

$$0 = v_0^2 + 2(-16)(200),$$

we obtain

$$v_0^2 = 6400.$$

Thus,

$$\boxed{v_0 = 80 \text{ ft/s.}}$$

In miles per hour,

$$\boxed{v_0 \approx 54.5 \text{ mi/h.}}$$

15. Convert 100 km/h to meters per second:

$$100 \text{ km/h} = \frac{250}{9} \text{ m/s.}$$

Using

$$0 = v_0^2 + 2a(80),$$

we get

$$a = -\frac{3125}{648} \text{ m/s}^2.$$

Therefore, the required deceleration has magnitude

$$\boxed{\frac{3125}{648}\ \mathrm{m/s^2} \approx 4.82\ \mathrm{m/s^2}.}$$

16. The maximum speed is

$$90\ \mathrm{mi/h} = 132\ \mathrm{ft/s}.$$

The time required to reach maximum speed is

$$t = \frac{132}{4} = 33\ \mathrm{s}.$$

The distance traveled while accelerating is

$$\frac{1}{2}(4)(33)^2 = 2178\ \mathrm{ft}.$$

The same distance is traveled while decelerating.

(a) If the train cruises at maximum speed for 15 minutes, then

$$d = 2178 + 132(900) + 2178.$$

Therefore,

$$\boxed{d = 123{,}156\ \mathrm{ft} = 23.325\ \mathrm{mi}.}$$

(b) The total available time is

$$15\ \mathrm{min} = 900\ \mathrm{s}.$$

After 33 seconds of acceleration and 33 seconds of deceleration, the cruising time is

$$900 - 66 = 834\ \mathrm{s}.$$

Thus,

$$d = 2178 + 132(834) + 2178.$$

Therefore,

$$\boxed{d = 114{,}444\ \mathrm{ft} = 21.675\ \mathrm{mi}.}$$

(c) The total distance is

$$45\ \mathrm{mi} = 237{,}600\ \mathrm{ft}.$$

The distance traveled while cruising is

$$237{,}600 - 2(2178) = 233{,}244 \text{ ft.}$$

The cruising time is

$$\frac{233{,}244}{132} = 1767 \text{ s.}$$

Therefore, the total time is

$$1767 + 33 + 33 = 1833 \text{ s.}$$

Hence,

$$\boxed{1833 \text{ s} = 30 \text{ min } 33 \text{ s.}}$$

Chapter 19: Areas and Distances

1. Here,

$$\Delta x = \frac{2-0}{4} = \frac{1}{2}.$$

Thus,

$$L_4 = \frac{1}{2}\left[f(0) + f\left(\frac{1}{2}\right) + f(1) + f\left(\frac{3}{2}\right)\right].$$

Therefore,

$$\boxed{L_4 = \frac{7}{4} = 1.75.}$$

2.

$$R_4 = \frac{1}{2}\left[f\left(\frac{1}{2}\right) + f(1) + f\left(\frac{3}{2}\right) + f(2)\right].$$

Therefore,

$$\boxed{R_4 = \frac{15}{4} = 3.75.}$$

3. Since $\Delta x = 1$,

$$L_4 = f(0) + f(1) + f(2) + f(3).$$

Therefore,

$$\boxed{L_4 = 4 + 3 + 2 + 1 = 10.}$$

4.

$$R_4 = f(1) + f(2) + f(3) + f(4).$$

Therefore,

$$\boxed{R_4 = 3 + 2 + 1 + 0 = 6.}$$

5. Since $\Delta x = 1$,

$$L_5 = f(0) + f(1) + f(2) + f(3) + f(4).$$

Therefore,

$$\boxed{L_5 = 1 + 2 + 3 + 4 + 5 = 15.}$$

6.

$$R_5 = f(1) + f(2) + f(3) + f(4) + f(5).$$

Therefore,

$$\boxed{R_5 = 2 + 3 + 4 + 5 + 6 = 20.}$$

7. Since $f(x) = x^3$ is increasing on $[0, 2]$, the left-endpoint approximation is an

$$\boxed{\text{underestimate.}}$$

8. Since $f(x) = x^3$ is increasing on $[0, 2]$, the right-endpoint approximation is an

$$\boxed{\text{overestimate.}}$$

9. Since $f(x) = 5 - x$ is decreasing, the left-endpoint approximation is an

$$\boxed{\text{overestimate.}}$$

10. Since $f(x) = 5 - x$ is decreasing, the right-endpoint approximation is an

$$\boxed{\text{underestimate.}}$$

11.

$$d = vt = 50(4).$$

Therefore,

$$\boxed{d = 200 \text{ miles.}}$$

12.

$$d = vt = 70(3).$$

Therefore,

$$\boxed{d = 210 \text{ miles.}}$$

13. The width of each time interval is

$$\Delta t = 1.$$

Using right endpoints,

$$d \approx 1\big[v(1) + v(2) + v(3) + v(4)\big].$$

Therefore,

$$\boxed{d \approx 18 + 20 + 21 + 24 = 83.}$$

Hint: For a right-endpoint approximation, use the velocity values at $t = 1, 2, 3, 4$.

14. Using left endpoints,

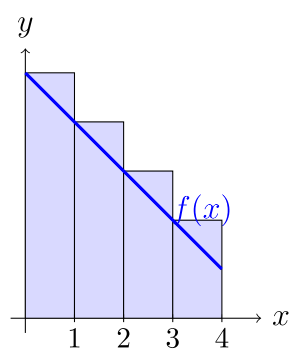
$$d \approx 1\big[v(0) + v(1) + v(2) + v(3)\big].$$

Therefore,

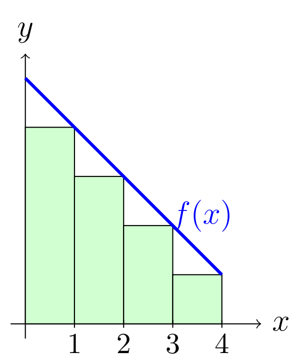
$$\boxed{d \approx 15 + 18 + 20 + 21 = 74.}$$

Hint: For a left-endpoint approximation, use the velocity values at $t = 0, 1, 2, 3$.

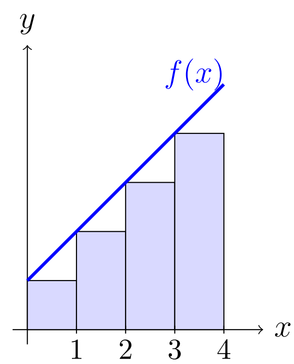
15. One possible sketch is shown below. Since the function is decreasing, the left-endpoint rectangles lie above the graph.



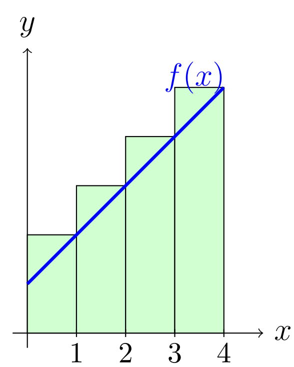
16. One possible sketch is shown below. Since the function is decreasing, the right-endpoint rectangles lie below the graph.



17. One possible sketch is shown below. Since the function is increasing, the left-endpoint rectangles lie below the graph.



18. One possible sketch is shown below. Since the function is increasing, the right-endpoint rectangles lie above the graph.



19. There are six intervals, each of width

$$\Delta t = 1.$$

Using the left endpoints $t = 0, 1, 2, 3, 4, 5$,

$$\begin{aligned} d &\approx 1\big(4.4 + 3.8 + 3.1 + 2.6 + 2.1 + 1.5\big) \\ &= 17.5. \end{aligned}$$

Therefore,

$$\boxed{d \approx 17.5 \text{ distance units.}}$$

Since the velocity is decreasing, this is an overestimate.

Hint: Do not use the value at $t = 6$ for a left-endpoint approximation.

20. There are six intervals, each of width

$$\Delta t = 1.$$

Using the right endpoints $t = 1, 2, 3, 4, 5, 6,$

$$\begin{aligned} d &\approx 1\big(1.5 + 2.1 + 2.8 + 3.3 + 3.7 + 4.2\big) \\ &= 17.6. \end{aligned}$$

Therefore,

$$\boxed{d \approx 17.6 \text{ distance units.}}$$

Since the velocity is increasing, this is an overestimate.

Hint: Do not use the value at $t = 0$ for a right-endpoint approximation.

Chapter 20: The Definite Integral

A. Evaluating Definite Integrals from the Definition

1.

$$\boxed{\int_0^3 x\,dx = \frac{9}{2}}$$

Hint: Use

$$\Delta x = \frac{3}{n}, \qquad x_i = \frac{3i}{n},$$

and

$$\int_0^3 x\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} x_i \Delta x.$$

2.

$$\boxed{\int_0^1 x^2\,dx = \frac{1}{3}}$$

Hint: Use

$$\Delta x = \frac{1}{n}, \qquad x_i = \frac{i}{n},$$

together with

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}.$$

3.

$$\boxed{\int_1^3 (2x+1)\,dx = 10}$$

Hint: Here

$$\Delta x = \frac{2}{n}, \qquad x_i = 1 + \frac{2i}{n}.$$

4.

$$\boxed{\int_0^2 (x^2+1)\,dx = \frac{14}{3}}$$

Hint: Use

$$\Delta x = \frac{2}{n}, \qquad x_i = \frac{2i}{n}.$$

B. Midpoint Rule

Recall that

$$M_n = \Delta x \sum_{i=1}^{n} f(\overline{x}_i),$$

where $\overline{x}_i$ is the midpoint of the i-th sub-interval.

5.

$$\boxed{M_4 = 21}$$

6.

$$\boxed{M_4 = 153}$$

7.

$$\boxed{M_6 = 30}$$

8.

$$\boxed{M_4 = \frac{\pi}{4}\left(\sqrt{2-\sqrt{2}} + \sqrt{2+\sqrt{2}}\right) \approx 2.0523}$$

Hint: The midpoints are

$$\frac{\pi}{8}, \quad \frac{3\pi}{8}, \quad \frac{5\pi}{8}, \quad \frac{7\pi}{8}.$$

9.

$$\boxed{M_4 = \frac{1}{2}\left(e^{1/4} + e^{3/4} + e^{5/4} + e^{7/4}\right) \approx 6.3230}$$

Hint: The midpoints are

$$\frac{1}{4}, \quad \frac{3}{4}, \quad \frac{5}{4}, \quad \frac{7}{4}.$$

C. Using Properties of Integrals

10.

$$\boxed{\frac{55}{2}}$$

11.

$$\boxed{\frac{129}{2}}$$

12.

$$\boxed{27}$$

13.

$$\boxed{0}$$

14.

$$\boxed{-\frac{124}{3}}$$

D. Geometric Evaluation

15.

$$\boxed{40}$$

16.

$$\boxed{18}$$

17.

$$\boxed{\frac{9\pi}{2}}$$

Hint: The graph is the upper half of a circle of radius 3.

18.

$$\boxed{8\pi}$$

Hint: The graph is the upper half of a circle of radius 4.

19.

$$\boxed{4}$$

20.

$$\boxed{4}$$

Hint: The graph forms a triangle with base 4 and height 2.

E. Net Area

21.

$$\boxed{\text{Positive}}$$

22.

$$\boxed{\text{Negative}}$$

23.

$$\boxed{\text{Zero}}$$

24.

$$\boxed{\text{Negative}}$$

F. Comparison Properties

25.

$$\boxed{5 \leq \int_2^7 f(x)\,dx \leq 20}$$

26.

$$\boxed{6 \leq \int_0^3 f(x)\,dx \leq 15}$$

27.

$$\boxed{-4 \leq \int_1^5 f(x)\,dx \leq 12}$$

G. True or False

28.

$$\boxed{\text{True}}$$

29.

$$\boxed{\text{False}}$$

30.

$$\boxed{\text{True}}$$

31.

$$\boxed{\text{True}}$$

32.

$$\boxed{\text{False}}$$

33.

$$\boxed{\text{True}}$$

H. Conceptual Questions

34. Area measures the total amount of space between the graph and the x-axis and is always nonnegative. Net area counts area above the x-axis positively and area below the x-axis negatively.

35. A definite integral has fixed endpoints and produces a single numerical value. An indefinite integral represents a family of antiderivatives:

$$\int f(x)\,dx = F(x) + C.$$

36. The quantity

$$\Delta x$$

is the width of each sub-interval (and each rectangle) in a Riemann sum.

37. Increasing the number of rectangles decreases their widths, so the rectangles follow the graph more closely, improving the approximation.

38. The variables x and t are dummy variables of integration. Therefore,

$$\int_a^b f(x)\,dx = \int_a^b f(t)\,dt.$$

39. Geometrically,

$$\int_a^b f(x)\,dx$$

represents the net signed area between the graph of f and the x-axis on $[a, b]$.

Chapter 21: The Fundamental Theorem of Calculus

Part A

1.

$$\boxed{x^4 - 3x + 1}$$

2.
$$\boxed{\sin x}$$

3.
$$\boxed{e^x}$$

4.
$$\boxed{\sqrt{x+4}}$$

Part B

1.
$$\boxed{21}$$

2.
$$\boxed{8}$$

3.
$$\boxed{2\sqrt{3}-\frac{2}{3}}$$

Hint: Rewrite
$$\sqrt{x}=x^{1/2}$$
and use the Power Rule.

4.
$$\boxed{2}$$

5.
$$\boxed{1}$$

Hint: An antiderivative of $\sec^2\theta$ is $\tan\theta$.

6.
$$\boxed{\frac{15}{32}}$$

Hint: Rewrite
$$\frac{1}{x^3}=x^{-3}.$$

7.
$$\boxed{-\frac{10}{3}}$$

Hint: Expand the integrand first:

$$(x+1)(x-2) = x^2 - x - 2.$$

8.

$$\boxed{20}$$

Hint: The change in position is

$$\int_0^4 v(t)\,dt.$$

9.

$$\boxed{\frac{21}{2}\text{ gallons}}$$

Hint: The total amount added is

$$\int_0^3 r(t)\,dt.$$

Part C

1.

$$\boxed{g'(x) = \sqrt{x + x^3}}$$

2.

$$\boxed{g'(x) = \cos(x^2)}$$

3.

$$\boxed{R'(y) = -y^3 \sin y}$$

Hint: Since the variable is the lower limit of integration, include a negative sign.

4.

$$\boxed{h'(x) = -\frac{1}{x^2}\sin^4\left(\frac{1}{x}\right)}$$

Hint: Use the Fundamental Theorem of Calculus together with the Chain Rule:

$$\frac{d}{dx}\left(\frac{1}{x}\right) = -\frac{1}{x^2}.$$

5.

$$\boxed{h'(x) = \frac{\sqrt{x}}{2(x^2+1)}}$$

Hint: Substitute $t = \sqrt{x}$ into the integrand, and then multiply by

$$\frac{d}{dx}(\sqrt{x}) = \frac{1}{2\sqrt{x}}.$$

6.

$$\boxed{\frac{dy}{dx} = \frac{3(3x+2)}{1+(3x+2)^3}}$$

Hint: Multiply the integrand evaluated at $3x+2$ by

$$\frac{d}{dx}(3x+2) = 3.$$

7.

$$\boxed{\frac{dy}{dx} = 2x\cos^2(x^2)}$$

Hint: Use the Chain Rule with the upper limit x^2.

8.

$$\boxed{\frac{dy}{dx} = -\frac{1}{2}\tan(\sqrt{x})}$$

Hint: The variable expression is the lower limit, so

$$\frac{dy}{dx} = -\left(\sqrt{x}\tan(\sqrt{x})\right)\left(\frac{1}{2\sqrt{x}}\right).$$

9.

$$\boxed{\frac{dy}{dx} = -\cos x\sqrt{1+\sin^2 x}}$$

Hint: The variable expression $\sin x$ is the lower limit.

10.

$$\boxed{g'(x) = 3\left(\frac{9x^2-1}{9x^2+1}\right) - 2\left(\frac{4x^2-1}{4x^2+1}\right)}$$

Hint: For variable upper and lower limits, use

$$\frac{d}{dx}\left(\int_{a(x)}^{b(x)} f(u)\,du\right) = f(b(x))b'(x) - f(a(x))a'(x).$$

11.

$$\boxed{g'(x) = 2(1+2x)\sin(1+2x) + 2(1-2x)\sin(1-2x)}$$

Hint: Apply the formula

$$\frac{d}{dx}\left(\int_{a(x)}^{b(x)} f(t)\,dt\right) = f(b(x))b'(x) - f(a(x))a'(x).$$

Be careful that

$$\frac{d}{dx}(1-2x) = -2.$$

12.

$$\boxed{h'(x) = 3x^2\cos(x^6) - \frac{\cos x}{2\sqrt{x}}}$$

Hint: Differentiate both variable limits:

$$h'(x) = \cos\left((x^3)^2\right)(3x^2) - \cos\left((\sqrt{x})^2\right)\left(\frac{1}{2\sqrt{x}}\right).$$

Part D

1.

$$\boxed{30}$$

2.

$$\boxed{-3}$$

3.

$$\boxed{4000\left(e^{1/2}-1\right)}$$

Hint: The change in population is

$$\int_0^{10} P'(t)\,dt.$$

4.

$$\boxed{F \text{ is increasing on } (-1,1).}$$

Hint: By the Fundamental Theorem of Calculus,

$$F'(x) = (1-x^2)\cos^2 x.$$

Since $\cos^2 x \geq 0$, determine where

$$1 - x^2 > 0.$$

5.

$$\boxed{F \text{ is concave upward wherever } f \text{ is increasing.}}$$

Indeed,

$$F'(x) = f(x)$$

and therefore

$$F''(x) = f'(x).$$

Thus, F is concave upward where

$$f'(x) > 0.$$

6.

$$\boxed{\frac{1}{4}}$$

Hint: Recognize the limit as the Riemann sum

$$\int_0^1 x^3\, dx.$$

7.

$$\boxed{1 - \cos 1}$$

Hint: Recognize the limit as the Riemann sum

$$\int_0^1 \sin x\, dx.$$

8.

$$\boxed{\frac{7}{3}}$$

Hint: Recognize the limit as the Riemann sum

$$\int_0^1 (1 + x)^2\, dx.$$

9.

$$\boxed{\frac{3x^2}{1+x^6} - \frac{2x}{1+x^4}}$$

Hint: Use

$$\frac{d}{dx}\left(\int_{a(x)}^{b(x)} f(t)\,dt\right) = f(b(x))b'(x) - f(a(x))a'(x).$$

10. Since

$$F'(x) = f(x),$$

the critical points of F occur where

$$f(x) = 0.$$

Therefore,

$$\boxed{x = 2 \text{ and } x = 5}$$

are critical points of F.

If f changes from positive to negative at one of these points, then F has a local maximum there. If f changes from negative to positive, then F has a local minimum there.

Chapter 22: Indefinite Integrals and the Net Change Theorem

Indefinite Integrals

1.

$$\boxed{\frac{10}{23}x^{23/10} + 2x^{7/2} + C}$$

Hint: Rewrite the decimal exponents as fractions:

$$1.3 = \frac{13}{10}, \qquad 2.5 = \frac{5}{2}.$$

2.

$$\boxed{\frac{4}{9}x^{9/4} + C}$$

Hint: Rewrite

$$\sqrt[4]{x^5} = x^{5/4}.$$

3.

$$\boxed{5x + \frac{2}{9}x^3 + \frac{3}{16}x^4 + C}$$

4.

$$\boxed{\frac{u^7}{7} - \frac{u^6}{3} - \frac{u^4}{4} + \frac{2u}{7} + C}$$

5.

$$\boxed{\frac{2}{3}u^3 + \frac{9}{2}u^2 + 4u + C}$$

Hint: Expand the product first:

$$(u+4)(2u+1) = 2u^2 + 9u + 4.$$

6.

$$\boxed{\theta + \tan\theta + C}$$

Hint: Use

$$\tan^2\theta = \sec^2\theta - 1.$$

7.

$$\boxed{\sec t + \tan t + C}$$

Hint: Expand the integrand:

$$\sec t(\sec t + \tan t) = \sec^2 t + \sec t \tan t.$$

8.

$$\boxed{-\cot t + \cos t + C}$$

Hint: Simplify the integrand:

$$\frac{1 - \sin^3 t}{\sin^2 t} = \csc^2 t - \sin t.$$

9.

$$\boxed{2\sin x + C}$$

Hint: Use
$$\sin(2x) = 2\sin x\cos x.$$

Definite Integrals

10.
$$\boxed{-\frac{10}{3}}$$

11.
$$\boxed{11}$$

12.
$$\boxed{\frac{21}{5}}$$

13.
$$\boxed{-429}$$

14.
$$\boxed{-2}$$

Hint: Expand the integrand before integrating:

$$(2x-3)(4x^2+1) = 8x^3 - 12x^2 + 2x - 3.$$

15.
$$\boxed{2\sqrt{5}}$$

Hint: Rewrite
$$\sqrt{\frac{5}{x}} = \sqrt{5}\,x^{-1/2}.$$

16.
$$\boxed{-\frac{9}{4}}$$

Hint: Rewrite the cube roots using rational exponents:

$$\frac{2}{\sqrt[3]{w}} - \sqrt[3]{w} = 2w^{-1/3} - w^{1/3}.$$

17.
$$\boxed{\frac{256}{15}}$$

Hint: Distribute $\sqrt{t}$:

$$\sqrt{t}(1+t) = t^{1/2} + t^{3/2}.$$

18.

$$\boxed{\sqrt{2}-1}$$

19.

$$\boxed{1+\frac{\pi}{4}}$$

Hint: Simplify the integrand:

$$\frac{1+\cos^2\theta}{\cos^2\theta} = \sec^2\theta + 1.$$

Conceptual Questions

20. It represents the child's change in weight from age 5 to age 10:

$$\boxed{w(10) - w(5) \text{ pounds.}}$$

21. It represents the net change in elevation of the trail between 3 miles and 5 miles from the start.

$$\boxed{\text{Net change in elevation from } x = 3 \text{ to } x = 5.}$$

22. The units are

$$(\text{newtons})(\text{meters}) = \boxed{\text{newton-meters}}.$$

23. Since $a(x)$ is measured in pounds per foot and x is measured in feet,

$$\boxed{\frac{da}{dx} \text{ has units of pounds per square foot}}$$

and

$$\boxed{\int_2^8 a(x)\,dx \text{ has units of pounds.}}$$

Chapter 23: The Substitution Rule

Part A

1.

$$\boxed{\frac{1}{2}\sin(2x) + C}$$

Hint: Let

$$u = 2x.$$

2.

$$\boxed{\frac{(2x^2+3)^5}{20} + C}$$

Hint: Let

$$u = 2x^2 + 3, \qquad du = 4x\,dx.$$

3.

$$\boxed{\frac{2}{9}(x^3+1)^{3/2} + C}$$

Hint: Let

$$u = x^3 + 1, \qquad du = 3x^2\,dx.$$

4.

$$\boxed{\frac{\sin^5\theta}{5} + C}$$

Hint: Let

$$u = \sin\theta, \qquad du = \cos\theta\,d\theta.$$

5.

$$\boxed{-\frac{1}{4(x^4-5)} + C}$$

Hint: Let

$$u = x^4 - 5, \qquad du = 4x^3\,dx.$$

6.

$$\boxed{\frac{1}{3}(2t+1)^{3/2} + C}$$

Hint: Let

$$u = 2t + 1.$$

7.

$$\boxed{-\frac{1}{3}(1-x^2)^{3/2}+C}$$

Hint: Let

$$u = 1 - x^2, \qquad du = -2x\,dx.$$

8.

$$\boxed{-\frac{1}{3}\cos(x^3)+C}$$

Hint: Let

$$u = x^3, \qquad du = 3x^2\,dx.$$

9.

$$\boxed{-\frac{(1-2x)^{10}}{20}+C}$$

Hint: Let

$$u = 1 - 2x, \qquad du = -2\,dx.$$

10.

$$\boxed{-\frac{2}{3}(1+\cos t)^{3/2}+C}$$

Hint: Let

$$u = 1 + \cos t, \qquad du = -\sin t\,dt.$$

Part B

1.

$$\boxed{\frac{\sin 2}{2}}$$

Hint: Let

$$u = 2x.$$

2.

$$\boxed{\frac{11^5 - 3^5}{20} = \frac{40202}{5}}$$

Hint: Let

$$u = 2x^2 + 3.$$

The new limits are

$$x = 0 \Rightarrow u = 3, \qquad x = 2 \Rightarrow u = 11.$$

3.

$$\boxed{\frac{2}{9}\left(2\sqrt{2} - 1\right)}$$

Hint: Let

$$u = x^3 + 1.$$

The new limits are

$$x = 0 \Rightarrow u = 1, \qquad x = 1 \Rightarrow u = 2.$$

4.

$$\boxed{\frac{1}{4}}$$

Hint: Let

$$u = \sin\theta.$$

5.

$$\boxed{\text{The integral diverges.}}$$

Hint: The denominator is zero when

$$x^4 - 5 = 0,$$

so

$$x = \sqrt[4]{5},$$

which lies in the interval $[1, 2]$. Thus, the integral is improper and does not converge.

6.

$$\boxed{\frac{7\sqrt{7} - 1}{3}}$$

Hint: Let

$$u = 2t + 1.$$

The new limits are

$$t = 0 \Rightarrow u = 1, \qquad t = 3 \Rightarrow u = 7.$$

7.

$$\boxed{\frac{1 - \cos 1}{3}}$$

Hint: Let

$$u = x^3.$$

8.

$$\boxed{0}$$

Hint: Let

$$u = x^2.$$

The new limits are

$$x = 0 \Rightarrow u = 0, \qquad x = \sqrt{\pi} \Rightarrow u = \pi.$$

9.

$$\boxed{\frac{2}{\sqrt{3}} - 1}$$

Hint: Let

$$u = \cos t, \qquad du = -\sin t\, dt.$$

10.

$$\boxed{\frac{2^{51} + 1}{153}}$$

Hint: Let

$$u = 3x - 1.$$

The new limits are

$$x = 0 \Rightarrow u = -1, \qquad x = 1 \Rightarrow u = 2.$$

Part C

1.

$$\boxed{\frac{(4x+1)^9}{36} + C}$$

Hint: Let

$$u = 4x + 1.$$

2.

$$\boxed{\frac{(x^2+9)^7}{14} + C}$$

Hint: Let

$$u = x^2 + 9, \qquad du = 2x\, dx.$$

3.
$$\boxed{\frac{1}{2}\ln(x^2+16)+C}$$

Hint: Let
$$u = x^2 + 16, \qquad du = 2x\,dx.$$

4.
$$\boxed{\frac{1}{3}\sin(x^3)+C}$$

Hint: Let
$$u = x^3, \qquad du = 3x^2\,dx.$$

5.
$$\boxed{-\frac{1}{5}\cos(5x)+C}$$

Hint: Let
$$u = 5x.$$

6.
$$\boxed{\frac{1}{4}\tan(4x)+C}$$

Hint: Let
$$u = 4x.$$

7.
$$\boxed{\frac{1}{7}e^{7x}+C}$$

Hint: Let
$$u = 7x.$$

8.
$$\boxed{\frac{2}{3}\sqrt{3x+2}+C}$$

Hint: Rewrite the integrand as
$$(3x+2)^{-1/2}$$

and let
$$u = 3x + 2.$$

9.

$$\boxed{78}$$

Hint: Let

$$u = x^2 + 1.$$

The new limits are

$$x = 0 \Rightarrow u = 1, \qquad x = 2 \Rightarrow u = 5.$$

10.

$$\boxed{\frac{2}{3}\left(2\sqrt{2} - 1\right)}$$

Hint: Let

$$u = 1 + \cos x, \qquad du = -\sin x\, dx.$$

The new limits are

$$x = 0 \Rightarrow u = 2, \qquad x = \frac{\pi}{2} \Rightarrow u = 1.$$

11.

$$\boxed{\frac{15}{136}}$$

Hint: Let

$$u = x^4 + 1, \qquad du = 4x^3\, dx.$$

The new limits are

$$x = 1 \Rightarrow u = 2, \qquad x = 2 \Rightarrow u = 17.$$

12.

$$\boxed{\frac{e-1}{3}}$$

Hint: Let

$$u = x^3, \qquad du = 3x^2\, dx.$$

13.

$$\boxed{\frac{284}{7}}$$

Hint: Since $x^6 + 1$ is even,

$$\int_{-2}^{2} (x^6 + 1)\, dx = 2 \int_{0}^{2} (x^6 + 1)\, dx.$$

14.

$$\boxed{0}$$

The function $x^7 + x$ is odd.

15.

$$\boxed{0}$$

The function $\sin x$ is odd.

16.

$$\boxed{2\sin 1}$$

Hint: Since $\cos x$ is even,

$$\int_{-1}^{1} \cos x \, dx = 2\int_{0}^{1} \cos x \, dx.$$

17.

$$\boxed{\text{Even}}$$

18.

$$\boxed{\text{Odd}}$$

Chapter 24: Areas Between Curves

1.

$$\boxed{\frac{39}{2}}$$

Hint: On $[-1, 2]$, $9 - x^2$ lies above $x + 1$, so use

$$A = \int_{-1}^{2} \left[(9 - x^2) - (x + 1)\right] \, dx.$$

2.

$$\boxed{\frac{3\pi^2}{8} - 1}$$

Hint: On $\left[\frac{\pi}{2}, \pi\right]$, $y = x$ lies above $y = \sin x$.

3.

$$\boxed{\frac{9}{2}}$$

Hint: The curves intersect at

$$x = 1 \qquad \text{and} \qquad x = 4.$$

4.

$$\boxed{36}$$

Hint: The curves intersect at

$$x = 0 \qquad \text{and} \qquad x = 6.$$

5.

$$\boxed{\frac{4}{3}}$$

Hint: The curves intersect at

$$x = -3 \qquad \text{and} \qquad x = 1.$$

6.

$$\boxed{72}$$

Hint: Solve

$$12 - x^2 = x^2 - 6$$

to obtain the limits $x = -3$ and $x = 3$.

7.

$$\boxed{\frac{16}{3}}$$

Hint: The curves intersect at

$$x = 0 \qquad \text{and} \qquad x = 2.$$

8.

$$\boxed{6\sqrt{3}}$$

Hint: On the given interval,

$$8\cos x \geq \sec^2 x.$$

Thus,

$$A = \int_{-\pi/3}^{\pi/3} \left(8\cos x - \sec^2 x\right)\, dx.$$

9.

$$\boxed{4\pi}$$

Hint: On $0 \leq x \leq 2\pi$,

$$2 - \cos x \geq \cos x.$$

10.

$$\boxed{\frac{32}{3}}$$

Hint: Integrate with respect to y. The curves intersect when

$$2y^2 = 4 + y^2,$$

so

$$y = -2, \qquad y = 2.$$

11.

$$\boxed{\frac{1}{6}}$$

Hint: Rewrite the line as

$$y = x - 1.$$

The intersections occur at

$$x = 1 \qquad \text{and} \qquad x = 2.$$

12.

$$\boxed{\frac{2}{3} + \frac{2}{\pi}}$$

Hint: The curves intersect at

$$x = -\frac{1}{2} \qquad \text{and} \qquad x = \frac{1}{2}.$$

On this interval, $\cos(\pi x)$ is the upper curve.

13.

$$\boxed{\frac{22}{15}}$$

Hint: Write the second curve as

$$x = 2 - y^2.$$

Integrate with respect to y from $y = 0$ to $y = 1$:

$$A = \int_0^1 \left[(2 - y^2) - y^4\right] \, dy.$$

14.

$$\boxed{2 - \frac{\pi}{2}}$$

Hint: The curves intersect at

$$x = 0, \qquad x = \frac{\pi}{2}, \qquad x = \pi.$$

The upper curve changes at $x = \frac{\pi}{2}$, so split the integral there.

15.

$$\boxed{39}$$

16.

$$\boxed{27}$$

Hint: Since the two regions lie on opposite sides of the curves,

$$|A_1 - A_2| = 15.$$

Given one area is 12, the other satisfies

$$A - 12 = 15.$$

17.

$$\boxed{\frac{3\sqrt{3}}{2} - 1}$$

Hint: Find where

$$\sin x = \cos 2x.$$

On $\left[0, \frac{\pi}{2}\right]$, the curves intersect at

$$x = \frac{\pi}{6}.$$

Therefore, split the absolute-value integral:

$$\int_0^{\pi/6} (\cos 2x - \sin x)\, dx + \int_{\pi/6}^{\pi/2} (\sin x - \cos 2x)\, dx.$$

18.

$$\boxed{\frac{44}{3} - 4\sqrt{6} - \frac{4\sqrt{2}}{3}}$$

Hint: Solve

$$\sqrt{x+2} = x.$$

The curves intersect at $x = 2$ on $[0, 4]$. Split the integral:

$$\int_0^2 \left(\sqrt{x+2} - x\right)\, dx + \int_2^4 \left(x - \sqrt{x+2}\right)\, dx.$$

19.

$$\boxed{\frac{1}{12}}$$

Hint: The tangent line to $y = x^2$ at $(1, 1)$ is

$$y = 2x - 1.$$

It intersects the x-axis at $x = \frac{1}{2}$. Split the area at $x = \frac{1}{2}$:

$$A = \int_0^{1/2} x^2\, dx + \int_{1/2}^1 \left[x^2 - (2x - 1)\right]\, dx.$$

20.

$$\boxed{b = 4^{2/3} = \sqrt[3]{16}}$$

Hint: Using horizontal slices, the width of the region at height y is

$$2\sqrt{y}.$$

Set

$$\int_0^b 2\sqrt{y}\, dy = \frac{1}{2} \int_0^4 2\sqrt{y}\, dy.$$

21.

$$\boxed{a = \frac{8}{5}}$$

Hint: Set the area from 1 to a equal to half the total area:

$$\int_1^a \frac{1}{x^2}\,dx = \frac{1}{2}\int_1^4 \frac{1}{x^2}\,dx.$$

Chapter 25: Volumes: Disk and Washer Methods

Basic Volume Problems

1.

$$V = \int_0^5 (3x+2)\,dx$$

$$V = \left[\frac{3}{2}x^2 + 2x\right]_0^5 = \frac{75}{2} + 10 = \frac{95}{2}.$$

$$\boxed{V = \frac{95}{2}}$$

2.

$$V = \int_0^3 x^2\,dx$$

$$V = \left[\frac{x^3}{3}\right]_0^3 = 9.$$

$$\boxed{V = 9}$$

3.

$$V = \int_{-3}^3 (9 - x^2)\,dx$$

Since $9 - x^2$ is an even function,

$$V = 2\int_0^3 (9 - x^2)\,dx.$$

Thus,

$$V = 2\left[9x - \frac{x^3}{3}\right]_0^3 = 2(27 - 9) = 36.$$

$$\boxed{V = 36}$$

Hint. Use symmetry, since $A(x) = 9 - x^2$ is an even function.

4.

$$V = \int_0^4 \sqrt{x}\, dx$$

$$V = \int_0^4 x^{1/2}\, dx = \left[\frac{2}{3}x^{3/2}\right]_0^4 = \frac{16}{3}.$$

$$\boxed{V = \frac{16}{3}}$$

5.

$$V = \int_0^1 e^x\, dx$$

$$V = [e^x]_0^1 = e - 1.$$

$$\boxed{V = e - 1}$$

The Disk Method

1.

$$\boxed{\frac{26\pi}{3}}$$

2.

$$\boxed{\frac{3\pi}{4}}$$

Hint: Using the Disk Method,

$$V = \pi \int_1^4 \left(\frac{1}{x}\right)^2\, dx.$$

3.

$$\boxed{8\pi}$$

Hint: The curve meets the x-axis when

$$\sqrt{x-1} = 0,$$

so the limits are $x = 1$ and $x = 5$. Use

$$V = \pi \int_1^5 \left(\sqrt{x-1}\right)^2\, dx.$$

4.

$$\boxed{162\pi}$$

Hint: Since the region is revolved about the y-axis, integrate with respect to y. The radius is

$$R(y) = 2\sqrt{y}.$$

Thus,

$$V = \pi \int_0^9 (2\sqrt{y})^2 \, dy.$$

5.

$$\boxed{\frac{256\pi}{5}}$$

Hint: Solve for x:

$$x = \frac{y^2}{2}.$$

The radius of each disk is

$$R(y) = \frac{y^2}{2},$$

so

$$V = \pi \int_0^4 \left(\frac{y^2}{2}\right)^2 \, dy.$$

The Washer Method

1.

$$\boxed{\frac{4\pi}{21}}$$

Hint: The curves intersect at

$$x = 0 \qquad \text{and} \qquad x = 1.$$

Using washers,

$$V = \pi \int_0^1 \left(x^2 - x^6\right) \, dx.$$

2.

$$\boxed{\frac{384\pi}{5}}$$

Hint: The curves intersect when

$$6 - x^2 = 2,$$

so the limits are $x = -2$ and $x = 2$. Use

$$V = \pi \int_{-2}^{2} \left[(6 - x^2)^2 - 2^2\right] \, dx.$$

3.

$$\boxed{\frac{11\pi}{30}}$$

Hint: In the first quadrant,

$$y = x^2 \qquad \text{and} \qquad y = \sqrt{x},$$

and the curves intersect at $x = 0$ and $x = 1$. Since the axis $y = 1$ lies above the region,

$$V = \pi \int_{0}^{1} \left[(1 - x^2)^2 - (1 - \sqrt{x})^2\right] \, dx.$$

4.

$$\boxed{\frac{471\pi}{14}}$$

Hint: The region extends from $x = 1$ to $x = 2$. About $y = -3$, the outer and inner radii are

$$R(x) = x^3 + 3, \qquad r(x) = 4.$$

Thus,

$$V = \pi \int_{1}^{2} \left[(x^3 + 3)^2 - 16\right] \, dx.$$

5.

$$\boxed{\frac{\pi}{2}\left(4\sqrt{2} - 3\right)}$$

Hint: On

$$0 \le x \le \frac{\pi}{4},$$

we have $\cos x \ge \sin x$. About $y = -1$,

$$R(x) = 1 + \cos x, \qquad r(x) = 1 + \sin x.$$

6.

$$\boxed{\frac{3\pi}{5}}$$

Hint: Since the axis of rotation is vertical, the Shell Method is convenient:

$$V = 2\pi \int_{0}^{1} (2 - x)x^3 \, dx.$$

7.

$$\boxed{\frac{10\sqrt{2}\pi}{3}}$$

Hint: Integrate with respect to y. The curves intersect when

$$y^2 = 1 - y^2,$$

so

$$y = \pm\frac{1}{\sqrt{2}}.$$

About $x = 3$,

$$R(y) = 3 - y^2, \qquad r(y) = 2 + y^2.$$

Therefore,

$$V = \pi \int_{-1/\sqrt{2}}^{1/\sqrt{2}} \left[(3 - y^2)^2 - (2 + y^2)^2\right]\, dy.$$

8.

$$\boxed{\pi\left(1 + \ln 2 - \frac{\pi}{4}\right)}$$

Hint: About $y = -1$,

$$R(x) = 1 + \tan x, \qquad r(x) = 1.$$

Thus,

$$V = \pi \int_0^{\pi/4} \left[(1 + \tan x)^2 - 1\right]\, dx.$$

Use

$$\tan^2 x = \sec^2 x - 1.$$

9.

$$\boxed{\frac{5\pi^2}{8}}$$

Hint: The axis $y = 1$ lies above the region. The outer and inner radii are

$$R(x) = 1, \qquad r(x) = 1 - \cos^2 x = \sin^2 x.$$

Hence,

$$V = \pi \int_{-\pi/2}^{\pi/2} \left(1 - \sin^4 x\right)\, dx.$$

Use symmetry and the identity

$$\sin^4 x = \frac{3 - 4\cos(2x) + \cos(4x)}{8}.$$

Chapter 26: Volumes: Cylindrical Shell Method

Part A

1.

$$\boxed{\frac{2\pi}{5}}$$

2.

$$\boxed{\frac{81\pi}{2}}$$

3.

$$\boxed{\frac{972\pi}{5}}$$

Hint: Using cylindrical shells,

$$V = 2\pi \int_0^9 x\sqrt{x}\, dx.$$

4.

$$\boxed{\frac{81\pi}{2}}$$

Hint: Since the solid is rotated about the x-axis, use horizontal shells:

$$V = 2\pi \int_0^3 y(y^2)\, dy.$$

Here, the shell radius is y, and the shell height is y^2.

5.

$$\boxed{\frac{81\pi}{2}}$$

Hint: Use horizontal shells with

$$\text{radius} = y, \qquad \text{height} = 9 - y^2.$$

Thus,

$$V = 2\pi \int_0^3 y(9 - y^2)\, dy.$$

6.

$$\boxed{\frac{\pi}{6}}$$

Hint: The curve meets the x-axis when

$$x - x^2 = 0,$$

so the limits are $x = 0$ and $x = 1$. Use

$$V = 2\pi \int_0^1 x(x - x^2)\, dx.$$

7.

$$\boxed{8\pi}$$

Hint: The region is symmetric about the y-axis. To avoid counting shells twice, use only

$$0 \le x \le 2.$$

Then

$$V = 2\pi \int_0^2 x(4 - x^2)\, dx.$$

8.

$$\boxed{128\pi}$$

Hint: The curves intersect when

$$y^2 = 16,$$

so $y = \pm 4$. Since rotation about the x-axis produces the same shells from positive and negative y-values, use only $0 \le y \le 4$:

$$V = 2\pi \int_0^4 y(16 - y^2)\, dy.$$

Part B

1.

$$\boxed{\frac{81\pi}{2}}$$

2.

$$\boxed{\frac{972\pi}{5}}$$

Hint: Use

$$V = 2\pi \int_0^9 x\sqrt{x}\,dx.$$

3.

$$\boxed{\frac{\pi}{2}}$$

Hint: The region lies between $x = 0$ and $x = 1$. The shell radius and height are

$$r(x) = 2 - x, \qquad h(x) = x - x^2.$$

Thus,

$$V = 2\pi \int_0^1 (2 - x)(x - x^2)\,dx.$$

4.

$$\boxed{8\pi}$$

Hint: The region is symmetric about the y-axis. Using only the right half prevents shells from being counted twice:

$$V = 2\pi \int_0^2 x(4 - x^2)\,dx.$$

5.

$$\boxed{\frac{875\pi}{6}}$$

Hint: The axis $y = -2$ passes through the region, so ordinary shells from $y = -3$ to $y = 3$ would overlap.

Let r be the distance from the axis. Then

$$0 \le r \le 5,$$

and the shell height is

$$9 - (r - 2)^2.$$

Therefore,

$$V = 2\pi \int_0^5 r\left[9 - (r - 2)^2\right]\,dr.$$

6. As written, the curves

$$y = x^3 \qquad \text{and} \qquad y = 8$$

do not enclose a bounded region.

$$\boxed{\text{No finite volume exists as stated.}}$$

If the y-axis is also intended as a boundary, then

$$V = 2\pi \int_0^2 x(8 - x^3)\, dx,$$

and

$$\boxed{V = \frac{96\pi}{5}}.$$

7.

$$\boxed{\frac{45\pi}{2}}$$

Hint: The axis $y = -1$ passes through the region, so overlapping shells must not be counted twice.

Let r denote the distance from $y = -1$. Then

$$0 \le r \le 3,$$

and the shell height is

$$4 - (r - 1)^2.$$

Thus,

$$V = 2\pi \int_0^3 r \left[4 - (r - 1)^2\right]\, dr.$$

8.

$$\boxed{2\pi^2}$$

Hint: Use

$$V = 2\pi \int_0^\pi x \sin x\, dx.$$

This integral requires integration by parts.

9.

$$\boxed{\pi^2 - 2\pi}$$

Hint: Use

$$V = 2\pi \int_0^{\pi/2} x \cos x\, dx.$$

This integral requires integration by parts.

10. As written, the curves

$$x = y^3 \qquad \text{and} \qquad x = 8$$

do not enclose a bounded region.

$$\boxed{\text{No finite volume exists as stated.}}$$

If the x-axis is also intended as a boundary, then

$$V = 2\pi \int_0^2 (y+1)(8 - y^3)\, dy,$$

and

$$\boxed{V = \frac{216\pi}{5}}.$$

Chapter 27: Work

Work

1.
$$\boxed{216\text{ J}}$$

2.
$$\boxed{55\text{ J}}$$

   Hint: Use
$$W = \int_0^5 (4x+1)\,dx.$$

3.
$$\boxed{0\text{ J}}$$

   Hint: Use
$$W = \int_1^2 \cos\left(\frac{\pi x}{3}\right)\,dx.$$
   An antiderivative is
$$\frac{3}{\pi}\sin\left(\frac{\pi x}{3}\right).$$

4.
$$\boxed{8\text{ J}}$$

   Hint: Interpret the work as the area under the force graph. The region consists of a triangle and a rectangle:
$$W = \frac{1}{2}(2)(2) + (3)(2).$$

5. Divide the interval $[a, b]$ into small subintervals of width
$$\Delta x.$$
   Over a sufficiently small subinterval, the force is approximately constant, so the work done is approximately
$$F(x_i^*)\Delta x.$$
   Adding the work over all subintervals gives
$$W \approx \sum_{i=1}^{n} F(x_i^*)\Delta x.$$
   As the number of subintervals increases and their widths approach zero, this Riemann

sum approaches the definite integral:

$$\boxed{W = \lim_{n\to\infty} \sum_{i=1}^{n} F(x_i^*)\Delta x = \int_a^b F(x)\,dx.}$$

Hook'e Law

1.

$$\boxed{4 \text{ J}}$$

Hint: Using Hooke's Law, $F(x) = kx$, so

$$W = \int_0^{0.4} 50x\,dx.$$

2.

$$\boxed{14.4 \text{ J}}$$

Hint: Use

$$W = \int_{0.1}^{0.5} 120x\,dx.$$

3.

$$\boxed{k = 300 \text{ N/m}}$$

Since $F = kx$,

$$18 = k(0.06).$$

4.

$$\boxed{1.2 \text{ J}}$$

Hint: The force stretches the spring

$$19 - 15 = 4 \text{ cm} = 0.04 \text{ m}.$$

Thus,

$$20 = k(0.04),$$

so

$$k = 500 \text{ N/m}.$$

The spring is stretched from

$$16 - 15 = 1 \text{ cm} = 0.01 \text{ m}$$

to

$$22 - 15 = 7 \text{ cm} = 0.07 \text{ m}.$$

Therefore,

$$W = \int_{0.01}^{0.07} 500x\,dx.$$

5.

$$\boxed{\frac{125}{48} \text{ ft} \cdot \text{lb}}$$

Hint: Convert the extensions to feet. Since

$$2 \text{ in} = \frac{1}{6} \text{ ft},$$

Hooke's Law gives

$$5 = k\left(\frac{1}{6}\right),$$

so

$$k = 30 \text{ lb/ft}.$$

The final extension is

$$5 \text{ in} = \frac{5}{12} \text{ ft}.$$

Thus,

$$W = \int_{0}^{5/12} 30x\,dx.$$

6.

$$\boxed{4 \text{ J}}$$

Hint: First find the spring constant:

$$30 = k(0.15),$$

so

$$k = 200 \text{ N/m}.$$

Then use

$$W = \int_{0.15}^{0.25} 200x\,dx.$$

Lifting Problems

1.

$$\boxed{1250 \text{ ft} \cdot \text{lb}}$$

Hint: A piece of rope x feet below the top must be lifted x feet. Thus,

$$W = \int_0^{25} 4x\,dx.$$

2.

$$\boxed{432 \text{ J}}$$

Hint: Use

$$W = \int_0^{12} 6x\,dx.$$

3.

$$\boxed{1200 \text{ ft} \cdot \text{lb}}$$

Hint: Use

$$W = \int_0^{40} 1.5x\,dx.$$

4.

$$\boxed{1600 \text{ ft} \cdot \text{lb}}$$

Hint: Add the work required to lift the bucket and the work required to lift the rope:

$$W = 50(20) + \int_0^{20} 3x\,dx.$$

5.

$$\boxed{375 \text{ ft} \cdot \text{lb}}$$

Hint: Since the water leaks at a constant rate, its weight after being lifted x feet is

$$30\left(1 - \frac{x}{25}\right).$$

Therefore,

$$W = \int_0^{25} 30\left(1 - \frac{x}{25}\right)\,dx.$$

6.

$$\boxed{\frac{375}{2} \text{ J}}$$

Hint: The lower half occupies the portion from 5 m to 10 m below the top. Thus,

$$W = \int_5^{10} 5x\,dx.$$

7.

$$\boxed{1600 \text{ ft} \cdot \text{lb}}$$

Hint: The lower 20 ft occupies the portion from 30 ft to 50 ft below the top. Therefore,

$$W = \int_{30}^{50} 2x\,dx.$$

8.

$$\boxed{1600 \text{ ft} \cdot \text{lb}}$$

Hint: Add the work required to lift the bucket and the rope:

$$W = 20(40) + \int_0^{40} x\,dx.$$

Pumping Liquids

1.

$$\boxed{19{,}968 \text{ ft} \cdot \text{lb}}$$

Hint: Let y measure height from the bottom of the tank. A horizontal layer has volume

$$dV = (8)(5)\,dy = 40\,dy$$

and must be lifted $4 - y$ feet. Thus,

$$W = 62.4 \int_0^4 40(4 - y)\,dy.$$

2.

$$\boxed{56{,}160 \text{ ft} \cdot \text{lb}}$$

Hint: Assuming the tank is full, a horizontal layer has volume

$$dV = (12)(6)\,dy = 72\,dy$$

and must be lifted $5 - y$ feet. Therefore,

$$W = 62.4 \int_0^5 72(5 - y)\,dy.$$

3.

$$\boxed{W = 62.4 \int_{-2}^{2} 16\sqrt{4 - y^2}\,(2 - y)\,dy}$$

Hint: Assume the cylindrical tank is horizontal and let its center be at $y = 0$. The width of a horizontal slice is

$$2\sqrt{4 - y^2}.$$

Since the tank is 8 ft long, the volume of the slice is

$$dV = 16\sqrt{4-y^2}\,dy.$$

The slice must be lifted $2-y$ feet.

4.

$$\boxed{W = 62.4\int_{-4}^{4} 30\sqrt{16-y^2}\,(4-y)\,dy}$$

Hint: Assuming the cylindrical tank is horizontal, the width of a horizontal slice is

$$2\sqrt{16-y^2}.$$

Multiplying by the tank length 15 gives

$$dV = 30\sqrt{16-y^2}\,dy.$$

5.

$$\boxed{404{,}352\ \text{ft}\cdot\text{lb}}$$

Hint: Since the top of the pool is at ground level, a horizontal layer at height y above the bottom must be lifted $6-y$ feet. Thus,

$$W = 62.4\int_0^6 (30)(12)(6-y)\,dy.$$

6.

$$\boxed{22{,}000\ \text{ft}\cdot\text{lb}}$$

Hint: A horizontal layer has volume

$$dV = (10)(4)\,dy = 40\,dy$$

and weight

$$55(40)\,dy.$$

It must be lifted $5-y$ feet. Thus,

$$W = 55\int_0^5 40(5-y)\,dy.$$

7. For a thin horizontal layer of liquid,

$$\text{weight} = (\text{weight density})(\text{volume of the layer}).$$

The small amount of work required to lift that layer is

$$dW = (\text{weight of the layer})(\text{distance the layer is lifted}).$$

Therefore, the total work is obtained by adding the work for all the layers:

$$\boxed{W = \int (\text{weight density})(\text{cross-sectional area})(\text{lifting distance})\, dy.}$$

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