MA 125
BUSINESS CALCULUS

FALL 2025

12.1 Local Extrema

September Monday 19^{th}, 2025.

Let $f(x)$ be a function and let $I = (a, b)$ be an open interval. Suppose $f'(x)$ exists on I.

(i). If $\boldsymbol{f'(x) > 0}$ for all x in I, then $f(x)$ is **increasing** on I.

(ii). If $\boldsymbol{f'(x) < 0}$ for all x in I, then $f(x)$ is **decreasing** on I.

(iii). If $\boldsymbol{f'(x) = 0}$ for all x in I, then $f(x)$ is a **constant** on I.

Note. The inequality $(x - a)(x - b) < 0$ is true only when $a < x < b$. Similarly, the inequality $(x - a)(x - b) > 0$ is true only when $x < a$ or $x > b$.

For an example, the inequality $(x - 1)(x - 2) < 0$ is true only when $1 < x < 2$, that is, only when x is in the interval $(1, 2)$.

Similarly, the inequality $(x - 1)(x - 2) > 0$ is true only when $x < 1$ or $x > 2$, that is, only when x is in the interval $(-\infty, 1)$ or in the interval $(2, \infty)$.

Example. Let $f(x) = 8x^3 - 18x^2 - 24x + 9$. Find the interval(s) on which $f(x)$ is increasing or decreasing.

Solution. We have that $f'(x) = 24x^2 - 36x - 24 = 12(2x^2 - 3x - 2) = 12(2x + 1)(x - 2)$. Thus,

$$\begin{aligned} f'(x) < 0 &\Leftrightarrow 12(2x+1)(x-2) < 0 \\ &\Leftrightarrow (2x+1)(x-2) < 0 \\ &\Leftrightarrow \left(x - -\frac{1}{2}\right)(x-2) < 0 \\ &\Leftrightarrow -\frac{1}{2} < x < 2. \end{aligned}$$

This says that $f'(x) < 0$ only when x is in the interval $\left(-\frac{1}{2}, 2\right)$. Thus, f is decreasing on $\left(-\frac{1}{2}, 2\right)$ and increasing on $\left(-\infty, -\frac{1}{2}\right)$ or $(2, \infty)$.

Critical Numbers.

Let $f(x)$ be a function. Let c be a number. If $f'(c) = 0$ or $f'(c)$ does not exist, then c is called a **critical number** of f. The pair $(c, f(c))$ is called a **critical point**.

Local (Relative) Extrema: Local (Relative) Maxima and Minima.

Let $f(x)$ be a function and let c be a number.

(i). f has a local (relative) maxima at $x = c$ if $f(x) \leq f(c)$ for all x near c.

(ii). f has a local (relative) minima at $x = c$ if $f(x) \geq f(c)$ for all x near c.

Note. If f has a local extremum (minimum or maximum) at $x = c$, then c is a critical number of f.

To determine whether a critical point is a local minimum or maximum, we have two tests. The first one is:

The First Derivative Test.

Let $f(x)$ be a function. Let $[a, b]$ be an interval. Suppose c be the only critical point of f in the interval (a, b). That is, $a < c < b$ and $f'(c) = 0$ or $f'(c)$ does not exist.

*(i). If $f'(a) > 0$ and $f'(b) < 0$, then there is a **local maximum** at $x = c$.*

*(ii). If $f'(a) < 0$ and $f'(b) > 0$, then there is a **local minimum** at $x = c$.*

*(iii). If both $f'(a)$ and $f'(b)$ have the same sign (plus or minus), then f has a **saddle point** (neither local minimum or local maximum) at $x = c$.*

Example. Find and determine the local minima and maxima of the function $f(x) = (x-2)^4$.

Solution. We have that $f'(x) = 4(x-2)^3$. Thus, by solving the equation $f'(x) = 0$ for x we get the critical number $x = 2$.

Choose $a = 0$ as the number less than 2 and choose $b = 3$ as the number bigger than $x = 2$. Then we get

- $f'(a) = f'(0) = 4(0-2)^3 = -32 < 0$ and
- $f'(b) = f'(3) = 3(3-2)^3 = 3 > 0$.

Therefore, by the first derivative test, $f(x)$ has a local minimum at $x = 2$.

Example. Find the dimensions of a rectangle with perimeter 1000 meters so that the area of the rectangle is a maximum.

Solution. Let l and w be the length and the width of the rectangle. Then,

$$1000 = l + w + l + w = 2l + 2w = 2(l + w).$$

This implies

$$l + w = 5000. \tag{1}$$

Let A be the area of the rectangle. Then $A = lw$. From (1) we have

$$A = l(5000 - l) = 5000l - l^2.$$

So, A can be thought of as a function of l. Then,

$A' = \frac{d}{dl}A = 5000 - 2l$. Thus, $A' = 0$ implies that $5000 - 2l = 0$, which implies $l = 2500$.

Now, take $a = 0$ as the smaller number and take $b = 5000$ as the bigger number. Then,

- $A'(0) = 5000 - 2 \cdot 0 = 5000 > 0$ and
- $A'(5000) = 5000 - 2 \cdot 5000 = 5000 - 10,000 = -5000 < 0$.

Thus, by the first derivative test, A has local maximum at $l = 2500$. So, the maximum area is

$$A_{max} = 2500(5000 - 2500) = 2500 \times 2500 = 6,250,000\ m^2.$$